5. $\sqrt{1+\cos 2x} = \sqrt{2\cos^2 x}$, which since $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, equals $\sqrt{2}\cos x$.

6. $f(x)$ has a corner at $x = 4$.

7. $\frac{d}{dx}(5x^{2/3}) = \frac{10}{3\sqrt[3]{x}}$ is undefined at $x = 0$. $f(x)$ has a cusp there.

8. $\frac{d}{dx}\left(\sqrt[5]{x+3}\right) = \frac{1}{5(x+3)^{4/5}}$ is undefined for $x = -3$. $f(x)$ has a vertical tangent there.

9. $\sqrt{x^2-4x+4} = |x-2|$ has a corner at $x = 2$.

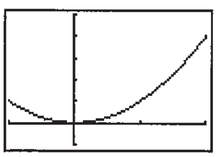
10. $\frac{d}{dx}(1+\sqrt[3]{\sin x}) = \frac{\cos x}{3(\sin x)^{2/3}}$ is undefined for $x = k\pi$, where k is any integer. $f(x)$ has vertical tangents at these values of x.

Section 7.4 Exercises

1. (a) $y' = 2x$, so

$$\text{Length} = \int_{-1}^{2} \sqrt{1+(2x)^2}\, dx = \int_{-1}^{2} \sqrt{1+4x^2}\, dx.$$

(b)

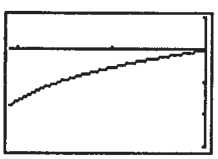


[−1, 2] by [−1, 5]

(c) Length ≈ 6.126

2. (a) $y' = \sec^2 x$, so Length $= \int_{-\pi/3}^{0} \sqrt{1+\sec^4 x}\, dx$.

(b)

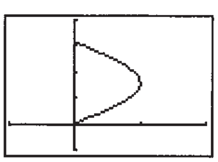


$\left[-\frac{\pi}{3}, 0\right]$ by [−3, 1]

(c) Length ≈ 2.057

3. (a) $x' = \cos y$, so Length $= \int_{0}^{\pi} \sqrt{1+\cos^2 y}\, dy$.

(b)

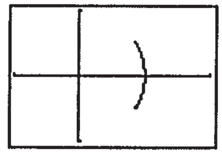


[−1, 2] by [−1, 4]

(c) Length ≈ 3.820

4. (a) $x' = -y(1-y^2)^{-1/2}$, so Length $= \int_{-1/2}^{1/2} \sqrt{1+\frac{y^2}{1-y^2}}\, dy$.

(b)



[−1, 2] by [−1, 1]

(c) Length ≈ 1.047

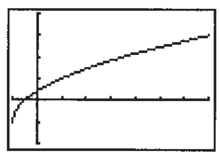
5. (a) $y^2+2y = 2x+1$, so

$y^2+2y+1 = (y+1)^2 = 2x+2$, and

$y = \sqrt{2x+2}-1$. Then $y' = \frac{1}{\sqrt{2x+2}}$, but NINT may fail using over the entire interval because y' is undefined at $x = -1$. So, use $x = \frac{(y+1)^2}{2} - 1$. Then $x' = y+1$ and

Length $= \int_{-1}^{3} \sqrt{1+(y+1)^2}\, dy$.

(b)



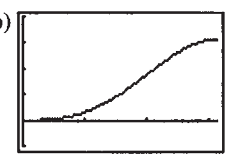
[−1, 7] by [−2, 4]

(c) Length ≈ 9.294

6. (a) $y' = \cos x + x\sin x - \cos x = x\sin x$, so

Length $= \int_{0}^{\pi} \sqrt{1+x^2\sin^2 x}\, dx$.

(b)

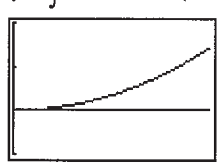


[0, π] by [−1, 4]

(c) Length ≈ 4.698

7. (a) $y' = \tan x$, so Length $= \int_{0}^{\pi/6} \sqrt{1+\tan^2 x}\, dx$.

(b) $y = \int \tan x\, dx = \ln(\sec x)$



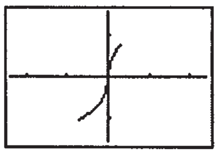
$\left[0, \frac{\pi}{6}\right]$ by [−0.1, 0.2]

(c) Length ≈ 0.549

8. (a) $x' = \sqrt{\sec^2 y - 1}$, so Length $= \int_{-\pi/3}^{\pi/4} \sec y\,dy$.

(b) $x' = \sqrt{\sec^2 y - 1} = |\tan y|$,

$$\text{so } x = \begin{cases} \ln(\cos y), & -\frac{\pi}{3} \le y \le 0 \\ -\ln(\cos y), & 0 < y \le \frac{\pi}{4} \end{cases}$$



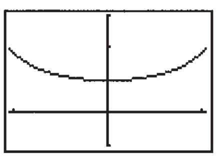
$[-2.4, 2.4]$ by $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) Length ≈ 2.198

9. (a) $y' = \sec x \tan x$, so

Length $= \int_{-\pi/3}^{\pi/3} \sqrt{1 + \sec^2 x \tan^2 x}\,dx$.

(b)

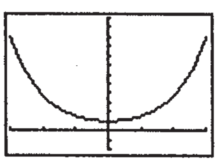


$\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ by $[-1, 3]$

(c) Length ≈ 3.139

10. $y' = \dfrac{(e^x - e^{-x})}{2}$, so Length $= \int_{-3}^{3} \sqrt{1 + \left(\dfrac{e^x - e^{-x}}{2}\right)^2}\,dx$.

(b)



$[-3, 3]$ by $[-2, 12]$

(c) Length ≈ 20.036

11. $y' = \frac{1}{2}(x^2+2)^{1/2}(2x) = x\sqrt{x^2+2}$, so the length is

$$\int_0^3 \sqrt{1 + \left(x\sqrt{x^2+2}\right)^2}\,dx = \int_0^3 \sqrt{x^4 + 2x^2 + 1}\,dx$$
$$= \int_0^3 (x^2+1)\,dx$$
$$= \left[\frac{1}{3}x^3 + x\right]_0^3 = 12.$$

12. $y' = \frac{3}{2}\sqrt{x}$, so the length is

$$\int_0^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\,dx = \int_0^4 \sqrt{1 + \frac{9x}{4}}\,dx$$
$$= \left[\frac{8}{27}\left(1 + \frac{9x}{4}\right)^{3/2}\right]_0^4$$
$$= \frac{80\sqrt{10} - 8}{27}.$$

13. $x' = y^2 - \dfrac{1}{4y^2}$, so the length is $\int_1^3 \sqrt{1 + \left(y^2 - \dfrac{1}{4y^2}\right)^2}\,dy$

$$= \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\,dy = \left[\frac{1}{3}y^3 - \frac{1}{4y}\right]_1^3 = \frac{53}{6}.$$

14. $x' = y^3 - \dfrac{1}{4y^3}$, so the length is

$$\int_1^2 \sqrt{1 + \left(y^3 - \frac{1}{4y^3}\right)^2}\,dy = \int_1^2 \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2}\,dy$$
$$= \left[\frac{1}{4}y^4 - \frac{1}{8y^2}\right]_1^2$$
$$= \frac{123}{32}.$$

15. $x' = \dfrac{y^2}{2} - \dfrac{1}{2y^2}$, so the length is

$$\int_1^2 \sqrt{1 + \left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2}\,dy = \int_1^2 \sqrt{\left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2}\,dy$$
$$= \left[\frac{1}{6}y^3 - \frac{1}{2y}\right]_1^2 = \frac{17}{12}.$$

16. $y' = x^2 + 2x + 1 - \dfrac{4}{(4x+4)^2} = (x+1)^2 - \dfrac{1}{4(x+1)^2}$

so the length is

$$\int_0^2 \sqrt{1 + \left((x+1)^2 - \frac{1}{4(x+1)^2}\right)^2}\,dx$$
$$= \int_0^2 \sqrt{\left((x+1)^2 + \frac{1}{4(x+1)^2}\right)^2}\,dx$$
$$= \left[\frac{1}{3}(x+1)^3 - \frac{1}{4(x+1)}\right]_0^2 = \frac{53}{6}.$$

17. $x' = \sqrt{\sec^4 y - 1}$, so the length is

$$\int_{-\pi/4}^{\pi/4} \sqrt{1 + (\sec^4 y - 1)}\,dy = \int_{-\pi/4}^{\pi/4} \sec^2 y\,dy$$
$$= \left[\tan y\right]_{-\pi/4}^{\pi/4} = 2.$$

18. $y' = \sqrt{3x^4 - 1}$, so the length is

$$\int_{-2}^{-1}\sqrt{1+(3x^4-1)}\,dx = \int_{-2}^{-1}\sqrt{3}x^2\,dx$$
$$= \sqrt{3}\left[\frac{1}{3}x^3\right]_{-2}^{-1} = \frac{7\sqrt{3}}{3}.$$

19. (a) $\left(\dfrac{dy}{dx}\right)^2$ corresponds to $\dfrac{1}{4x}$ here, so take $\dfrac{dy}{dx}$ as $\dfrac{1}{2\sqrt{x}}$.

Then $y = \sqrt{x} + C$, and, since (1, 1) lies on the curve, $C = 0$. So $y = \sqrt{x}$.

(b) Two. We know the value of the function at one value of x, and we know the square of the derivative. We can also let $\dfrac{dy}{dx} = -\dfrac{1}{2\sqrt{x}}$, which gives the solution $y = 2 - \sqrt{x}$.

20. (a) $\left(\dfrac{dx}{dy}\right)^2$ corresponds to $\dfrac{1}{y^4}$ here, so take $\dfrac{dx}{dy}$ as $\dfrac{1}{y^2}$.

Then $x = -\dfrac{1}{y} + C$ and, since (0, 1) lies on the curve, $C = 1$. So $y = \dfrac{1}{1-x}$.

(b) Two. We know the value of the function at one value of x, and we know the square of the derivative. We can also let $\dfrac{dx}{dy} = -\dfrac{1}{y^2}$, which gives the solution $y = \dfrac{1}{x+1}$.

21. $y' = \sqrt{\cos 2x}$, so the length is

$$\int_0^{\pi/4}\sqrt{1+\cos 2x}\,dx = \int_0^{\pi/4}\sqrt{2\cos^2 x}\,dx$$
$$= \sqrt{2}\Big[\sin x\Big]_0^{\pi/4} = 1.$$

22. $y' = -(1-x^{2/3})^{1/2}x^{-1/3}$, so the length is

$$8\int_{\sqrt{2}/4}^{1}\sqrt{1+(1-x^{2/3})\,x^{-2/3}}\,dx = 8\int_{\sqrt{2}/4}^{1}\sqrt{x^{-2/3}}\,dx$$
$$= 8\int_{\sqrt{2}/4}^{1} x^{-1/3}\,dx$$
$$= 8\left[\frac{3}{2}x^{2/3}\right]_{\sqrt{2}/4}^{1}$$
$$= 8\left[\frac{3}{2} - \frac{3}{2}\left(\frac{1}{2}\right)\right] = 6.$$

23. Find the length of the curve $y = \sin\dfrac{3\pi}{20}x$ for $0 \le x \le 20$.

$y' = \dfrac{3\pi}{20}\cos\dfrac{3\pi}{20}x$, so the length is

$$\int_0^{20}\sqrt{1+\left(\frac{3\pi}{20}\cos\frac{3\pi}{20}x\right)^2}\,dx,$$ which evaluates, using NINT, to ≈ 21.07 inches.

24. The area is 300 times the length of the arch.

$y' = -\left(\dfrac{\pi}{2}\right)\sin\left(\dfrac{\pi x}{50}\right)$, so the length is

$$\int_{-25}^{25}\sqrt{1+\left(\frac{\pi}{2}\right)^2\sin^2\left(\frac{\pi x}{50}\right)}\,dx,$$ which evaluates, using NINT, to ≈ 73.185. Multiply that by 300, then by \$1.75 to obtain the cost (rounded to the nearest dollar): \$38,422

25. $f'(x) = \dfrac{1}{3}x^{-2/3} + \dfrac{2}{3}x^{-1/3}$, but NINT fails on

$\int_0^2\sqrt{1+(f'(x))^2}\,dx$ because of the undefined slope at $x = 0$. So, instead solve for x in terms of y using the quadratic formula. $(x^{1/3})^2 + x^{1/3} - y = 0$, and

$x^{1/3} = \dfrac{-1\pm\sqrt{1+4y}}{2}$. Using the positive values,

$x = \dfrac{1}{8}\left(\sqrt{1+4y}-1\right)^3$. Then,

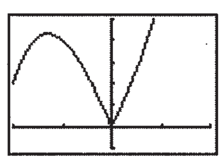
$x' = \dfrac{3}{8}\left(\sqrt{1+4y}-1\right)^2\left(\dfrac{2}{\sqrt{1+4y}}\right)$, and

$$\int_0^{2^{1/3}+2^{2/3}}\sqrt{1+(x')^2}\,dy \approx 3.6142.$$

26. $f'(x) = \dfrac{(4x^2+1)-(8x^2-8x)}{(4x^2+1)^2} = -\dfrac{4x^2-8x-1}{(4x+1)^2}$, so the

length is $\displaystyle\int_{-1/2}^{1}\sqrt{1+\left(\frac{4x^2-8x-1}{\left(4x^2+1\right)^2}\right)^2}\,dx$ which evaluates, using NINT, to ≈ 2.1089.

27. There is a corner at $x = 0$:



[–2, 2] by [–1, 5]

Break the function into two smooth segments:

$$y = \begin{cases} x^3 - 5x, & -2 \le x \le 0 \\ x^3 + 5x, & 0 < x \le 1 \end{cases} \quad \text{and}$$

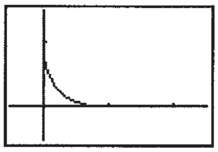
$$y' = \begin{cases} 3x^2 - 5, & -2 \le x < 0 \\ 3x^2 + 5, & 0 < x \le 1 \end{cases}$$

The length is $\int_{-2}^{1}\sqrt{1+(y')^2}\,dy$

$= \int_{-2}^{0}\sqrt{1+(3x^2-5)^2}\,dx + \int_0^1\sqrt{1+(3x^2+5)^2}\,dx$,

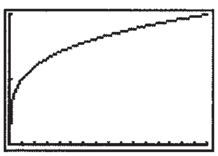
which evaluates, using NINT for each part, to ≈ 13.132.

28. $y=(1-\sqrt{x})^2, 0\le x\le 1$



[–0.5, 2.5] by [–0.5, 1.5]

$y'=\dfrac{\sqrt{x}-1}{\sqrt{x}}$, but NINT may fail using y' over the entire interval because y' is undefined at $x=0$. So, split the curve into two equal segments by solving $\sqrt{x}+\sqrt{y}=1$ with $y=x$: $x=\dfrac{1}{4}$. The total length is $2\int_{1/4}^{1}\sqrt{1+\left(\dfrac{\sqrt{x}-1}{\sqrt{x}}\right)^2}\,dx$, which evaluates, using NINT, to ≈ 1.623.

29. 

[0, 16] by [0, 2]

$y'=\dfrac{1}{4}x^{-3/4}$, but NINT may fail using y' over the entire interval, because y' is undefined at $x=0$. So, use $x=y^4,\ 0\le y\le 2$: $x'=4y^3$ and the length is $\int_0^2\sqrt{1+(4y^3)^2}\,dy$, which evaluates, using NINT, to ≈ 16.647.

30. Horizontal line segments will not work because the limit of the sum $\Sigma\Delta x_k$, as the norm of the partition goes to zero, will always be the length $(b-a)$ of the interval (a, b).

31. No; the curve can be indefinitely long. Consider, for example, the curve $\dfrac{1}{3}\sin\left(\dfrac{1}{x}\right)+0.5$ for $0<x<1$.

32. False. The function must be differentiable.

33. True, by definition.

34. D. $\dfrac{dy}{dx}=-2\sin(2x)$

$$\int_0^{\pi/4}\sqrt{1+(-2\sin(2x))^2}\,dx\approx 1.318$$

35. C. $\dfrac{dx}{dy}=3y^2$

$$\int_{-2}^{2}\sqrt{1+(3y^2)^2}\,dy=\int_{-2}^{2}\sqrt{1+9y^4}\,dy$$

36. B. $\dfrac{dy}{dx}=\sqrt{x}$

$$\int_0^8\sqrt{1+\sqrt{x}^2}\,dx=\frac{52}{3}$$

37. A. $\dfrac{dx}{dy}=\dfrac{3}{2}y^{1/2}$

$$2\int_0^1\sqrt{1+\left(\frac{3}{2}y^{1/2}\right)^2}\,dy=2\int_0^1\sqrt{1+\frac{9}{4}y}\,dy$$

38. For track 1: $y_1=0$ at $x=\pm 10\sqrt{5}\approx\pm 22.3607$, and $y_1{}'=\dfrac{-0.2x}{\sqrt{100-0.2x^2}}$. NINT fails to evaluate $\int_{-10\sqrt{5}}^{10\sqrt{5}}\sqrt{1+(y_1{}')^2}\,dx$ because of the undefined slope at the limits, so use the track's symmetry, and "back away" from the upper limit a little, and find $2\int_0^{22.36}\sqrt{1+(y_1{}')^2}\,dx\approx 52.548$. Then, approximating the last little stretch at each end by a straight vertical line, add $2\sqrt{100-0.2(22.36)^2}\approx 0.156$ to get the total length of track 1 as ≈ 52.704. Using a similar strategy, find the length of the *right half* of track 2 to be ≈ 32.274. Store the unrounded value as A. Now enter $Y_1=52.704$ and

$$Y_2=A+\text{NINT}\left(\sqrt{1+\left(\frac{-0.2t}{\sqrt{150-0.2t^2}}\right)^2},\ t,\ x,\ 0\right)$$ and graph

in a $[-30,0]$ by $[0,60]$ window to see the effect of the x-coordinate of the lane-2 starting position on the length of lane 2. (Be patient!) Solve graphically to find the intersection at $x\approx -19.909$, which leads to starting point coordinates (–19.909, 8.410).

39. (a) The fin is the hypotenuse of a right triangle with leg lengths Δx_k and $\left.\dfrac{df}{dx}\right|_{x=x_{k-1}}\Delta x_k=f'(x_{k-1})\Delta x_k$.

(b)
$$\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{(\Delta x_k)^2+(f'(x_{k-1})\Delta x_k)^2}$$
$$=\lim_{n\to\infty}\sum_{k=1}^{n}\Delta x_k\sqrt{1+(f'(x_{k-1}))^2}$$
$$=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

40. Yes. Any curve of the form $y=\pm x+c$, where c is a constant, has constant slope ± 1, so that

$$\int_0^a\sqrt{1+(y')^2}\,dx=\int_0^a\sqrt{2}\,dx=a\sqrt{2}.$$