Section 7.3 Volumes (pp. 399–411)

Exploration 1 Volume by Cylindrical Shells

1. Its height is $f(x_k) = 3x_k - x_k^2$.

2. Unrolling the cylinder, the circumference becomes one dimension of a rectangle, and the height becomes the other. The thickness Δx is the third dimension of a slab with dimensions $2\pi(x_k + 1)$ by $3x_k - x_k^2$ by Δx. The volume is obtained by multiplying the dimensions together.

3. The limit is the definite integral $\int_0^3 2\pi(x+1)(3x-x^2)\,dx$

4. $\dfrac{45\pi}{2}$

Exploration 2 Surface Area

1. $\int_a^b 2\pi y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx$

The limit will exist if f and f' are continuous on the interval $[a, b]$.

2. $y=\sin x$, so $\dfrac{dy}{dx}=\cos x$ and

$$\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\int_0^\pi 2\pi \sin x\sqrt{1+\cos^2 x}\,dx \approx 14.424.$$

3. $y=\sqrt{x}$, so $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}}$ and $\int_0^4 2\pi\sqrt{x}\sqrt{1+\left(\dfrac{1}{2\sqrt{x}}\right)^2}\,dx \approx 36.177.$

Quick Review 7.3

1. x^2

2. $s=\dfrac{x}{\sqrt{2}}$, so Area $= s^2 = \dfrac{x^2}{2}$.

3. $\dfrac{1}{2}\pi r^2$ or $\dfrac{\pi x^2}{2}$

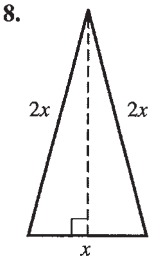
4. $\dfrac{1}{2}\pi\left(\dfrac{d}{2}\right)^2$ or $\dfrac{\pi x^2}{8}$

5. $b = x$ and $h=\dfrac{\sqrt{3}}{2}x$, so Area $=\dfrac{1}{2}bh=\dfrac{\sqrt{3}}{4}x^2$.

6. $b = h = x$, so Area $=\dfrac{1}{2}bh=\dfrac{x^2}{2}$.

7. $b = h = \dfrac{x}{\sqrt{2}}$, so Area $=\dfrac{1}{2}bh=\dfrac{x^2}{4}$.

8.



$b = x$ and $h=\sqrt{(2x^2)-\left(\dfrac{1}{2}x\right)^2}=\dfrac{\sqrt{15}}{2}x$, so

Area $=\dfrac{1}{2}bh=\dfrac{\sqrt{15}}{4}x^2$.

9. This is a 3-4-5 right triangle.

$b = 4x$, $h = 3x$, and Area $=\dfrac{1}{2}bh = 6x^2$.

10. The hexagon contains six equilateral triangles with sides of length x, so from Exercise 5, Area $= 6\left(\dfrac{\sqrt{3}}{4}x^2\right)+\dfrac{3\sqrt{3}}{2}x^2$.

Section 7.3 Exercises

1. In each case, the width of the cross section is $w = 2\sqrt{1-x^2}$.

(a) $A=\pi r^2$, where $r=\dfrac{w}{2}$, so $A(x)=\pi\left(\dfrac{w}{2}\right)^2=\pi(1-x^2)$.

(b) $A=s^2$, where $s=w$, so $A(x)=w^2=4(1-x^2)$.

(c) $A=s^2$, where $s=\dfrac{w}{\sqrt{2}}$, so $A(x)=\left(\dfrac{w}{\sqrt{2}}\right)^2=2(1-x^2)$.

(d) $A=\dfrac{\sqrt{3}}{4}w^2$ (see Quick Review Exercise 5), so

$A(x)=\dfrac{\sqrt{3}}{4}(2\sqrt{1-x^2})^2=\sqrt{3}(1-x^2)$.

2. In each case, the width of the cross section is $w = 2\sqrt{x}$.

(a) $A=\pi r^2$, where $r=\dfrac{w}{2}$, so $A(x)=\pi\left(\dfrac{w}{2}\right)^2=\pi x$.

(b) $A=s^2$, where $s=w$, so $A(x)=w^2=4x$.

(c) $A=s^2$, where $s=\dfrac{w}{\sqrt{2}}$, so $A(x)=\left(\dfrac{w}{\sqrt{2}}\right)^2=2x$.

(d) $A=\dfrac{\sqrt{3}}{4}w^2$ (see Quick Review Exercise 5), so

$A(x)=\dfrac{\sqrt{3}}{4}(2\sqrt{x})^2=\sqrt{3}x$.

3. A cross section has width $w = 2\sqrt{x}$ and area

$A(x) = s^2 = \left(\frac{w}{\sqrt{2}}\right)^2 = 2x$. The volume is

$\int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16.$

4. A cross section has width $w = (2 - x^2) - x^2 = 2 - 2x^2$ and

area $A(x) = \pi r^2 = \pi\left(\frac{w}{2}\right)^2 = \pi(1 - x^2)^2$. The volume is

$$\int_{-1}^{1} (1 - x^2)^2\,dx = \pi\int_{-1}^{1} (x^4 - 2x^2 + 1)\,dx$$
$$= \pi\left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_{-1}^{1} = \frac{16}{15}\pi.$$

5. The cross section has width $w = 2\sqrt{1 - x^2}$ and area

$A(x) = s^2 = w^2 = 4(1 - x^2)$. The volume is

$$\int_{-1}^{1} 4(1 - x^2)\,dx = 4\int_{-1}^{1} (1 - x^2)\,dx = 4\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{16}{3}.$$

6. A cross section has width $w = 2\sqrt{1 - x^2}$ and area

$A(x) = s^2 = \left(\frac{w}{\sqrt{2}}\right)^2 = 2(1 - x^2)$. The volume is

$$\int_{-1}^{1} 2(1 - x^2)\,dx = 2\int_{-1}^{1} (1 - x^2)\,dx = 2\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{8}{3}.$$

7. The solid is a right circular cone of radius 1 and height 2.

$V = \frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi \cdot 1^2) \cdot 2 = \frac{2}{3}\pi$

Using integration: A cross section has radius $\left(1 - \frac{1}{2}x\right)$

and area $A(x) = \pi\left(1 - \frac{1}{2}x\right)^2$. The volume is

$$V = \int_0^2 \pi\left(1 - \frac{1}{2}x\right)^2 dx = \pi\int_0^2\left(\frac{x^2}{4} - x + 1\right)dx$$
$$= \pi\left[\frac{x^3}{12} - \frac{x^2}{2} + x\right]_0^2 = \frac{2}{3}\pi.$$

8. The solid is a right circular cone of radius 3 and height 2.

$V = \frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi \cdot 3^2) \cdot 2 = 6\pi$

Using integration: A cross section has radius $\left(\frac{3y}{2}\right)$

and area $A(y) = \pi\left(\frac{3y}{2}\right)^2$. The volume is

$$V = \int_0^2 \pi\left(\frac{3y}{2}\right)^2 dy = \frac{9}{4}\pi\int_0^2 y^2\,dy$$
$$= \frac{9}{4}\pi\left[\frac{y^3}{3}\right]_0^2 = 6\pi.$$

9. A cross section has radius $r = \tan\left(\frac{\pi}{4}y\right)$ and area

$A(y) = \pi r^2 = \pi\tan^2\left(\frac{\pi}{4}y\right)$. The volume is

$$\int_0^1 \pi\tan^2\left(\frac{\pi}{4}y\right)dy = \pi\left[\frac{4}{\pi}\tan\left(\frac{\pi}{4}y\right) - y\right]_0^1$$
$$= \pi\left(\frac{4}{\pi} - 1\right)$$
$$= 4 - \pi.$$

10. A cross section has radius $r = \sin x \cos x$ and area

$A(x) = \pi r^2 = \pi\sin^2 x\cos^2 x$. The shaded region extends

from $x = 0$ to where $\sin x \cos x$ drops back to 0, i.e., where

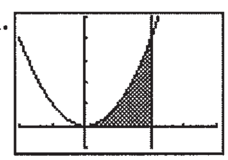
$x = \frac{\pi}{2}$. Now, since $\cos 2x = 2\cos^2 x - 1$, we know

$\cos^2 x = \frac{1 + \cos 2x}{2}$ and since $\cos 2x = 1 - 2\sin^2 x$, we

know $\sin^2 x = \frac{1 - \cos 2x}{2}$.

$$\int_0^{\pi/2} \pi\sin^2 x\cos^2 x\,dx$$
$$= \pi\int_0^{\pi/2}\frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2}dx$$
$$= \frac{\pi}{4}\int_0^{\pi/2}(1 - \cos^2 2x)\,dx = \frac{\pi}{4}\int_0^{\pi/2}\sin^2 2x\,dx$$
$$= \frac{\pi}{4}\int_0^{\pi/2}\frac{1 - \cos 4x}{2}dx = \frac{\pi}{8}\int_0^{\pi/2}(1 - \cos 4x)\,dx$$
$$= \frac{\pi}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi^2}{16}.$$

11.



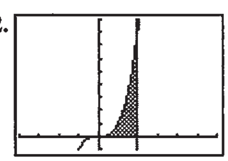
[–2, 4] by [–1, 5]

A cross section has radius $r = x^2$ and area

$A(x) = \pi r^2 = \pi x^4$. The volume is

$$\int_0^2 \pi x^4\,dx = \pi\left[\frac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}.$$

12.

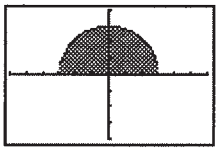


[–4, 6] by [–1, 9]

A cross section has radius $r = x^3$ and area

$A(x) = \pi r^2 = \pi x^6$. The volume is

$$\int_0^2 \pi x^6\,dx = \pi\left[\frac{1}{7}x^7\right]_0^2 = \frac{128\pi}{7}.$$

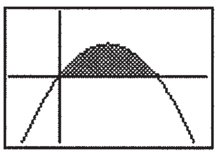
13. 

[−6, 6] by [−4, 4]

The solid is a sphere of radius $r = 3$. The volume is $\frac{4}{3}\pi r^3 = 36\pi.$

Using integration: A cross section has radius $\sqrt{9-x^2}$ and area $A(y) = \pi\left(\sqrt{9-x^2}\right)^2$. The volume is

$$V = \int_{-3}^{3} \pi\left(\sqrt{9-x^2}\right)^2 dx = \pi\int_{-3}^{3}(9-x^2)dx$$
$$= \pi\left[9x - \frac{1}{3}x^3\right]_{-3}^{3} = 36\pi.$$

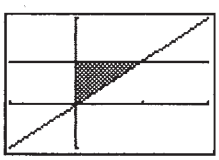
14. 

[−0.5, 1.5] by [−0.5, 0.5]

The parabola crosses the line $y = 0$ when $x - x^2 = x(1-x) = 0$, i.e., when $x = 0$ or $x = 1$. A cross section has radius $r = x - x^2$ and area $A(x) = \pi r^2 = \pi(x-x^2)^2 = \pi(x^2 - 2x^3 + x^4).$

The volume is

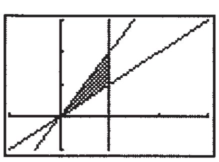
$$\int_0^1 \pi\left(x^2 - 2x^3 + x^4\right)dx = \pi\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1 = \frac{\pi}{30.}$$

15. 

[−1, 2] by [−1, 2]

Use cylindrical shells: A shell has radius y and height y.

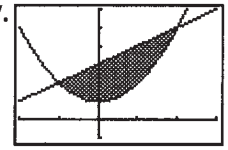
The volume is $\int_0^1 2\pi(y)(y)\,dy = 2\pi\left[\frac{1}{3}y^3\right]_0^1 = \frac{2}{3}\pi.$

16. 

[−1, 3] by [−1, 3]

Use washer cross sections: A washer has inner radius $r = x$, outer radius $R = 2x$, and area $A(x) = \pi(R^2 - r^2) = 3\pi x^2$.

The volume is $\int_0^1 3\pi x^2\,dx = 3\pi\left[\frac{1}{3}x^3\right]_0^1 = \pi.$

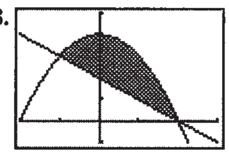
17. 

[−2, 3] by [−1, 6]

The curves intersect when $x^2+1 = x+3$, which is when $x^2 - x - 2 = (x-2)(x+1) = 0$, i.e., when $x = -1$ or $x = 2$. Use washer cross sections: a washer has inner radius $r = x^2+1$, outer radius $R = x+3$, and area

$$A(x) = \pi(R^2 - r^2)$$
$$= \pi\left[(x+3)^2 - (x^2+1)^2\right]$$
$$= \pi(-x^4 - x^2 + 6x + 8).$$ The volume is

$$\int_{-1}^{2} \pi(-x^4 - x^2 + 6x + 8)\,dx$$
$$= \pi\left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 + 3x^2 + 8x\right]_{-1}^{2}$$
$$= \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + 12 + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + 3 - 8\right)\right] = \frac{117\pi}{5}.$$

18. 

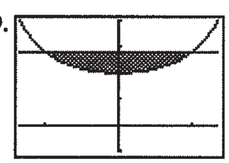
[−2, 3] by [−1, 5]

The curves intersect when $4 - x^2 = 2 - x$, which is when $x^2 - x - 2 = (x-2)(x+1) = 0$, i.e., when $x = -1$ or $x = 2$. Use washer cross sections: a washer has inner radius $r = 2 - x$, outer radius $R = 4 - x^2$, and area

$$A(x) = \pi(R^2 - r^2)$$
$$= \pi\left[(4-x^2)^2 - (2-x)^2\right]$$
$$= \pi(12 + 4x - 9x^2 + x^4).$$

The volume is

$$\int_{-1}^{2} \pi(12 + 4x - 9x^2 + x^4)\,dx$$
$$= \pi\left[12x + 2x^2 - 3x^3 + \frac{1}{5}x^5\right]_{-1}^{2}$$
$$\pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right] = \frac{108\pi}{5}.$$

19. 

$\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ by [−0.5, 2]

Use washer cross sections: a washer has inner radius $r = \sec x$. outer radius $R = \sqrt{2}$, and area

$$A(x) = \pi(R^2 - r^2) = \pi(2 - \sec^2 x).$$

19. Continued

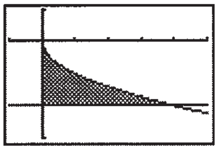
The volume is

$$\int_{-\pi/4}^{\pi/4} \pi(2-\sec^2 x)\,dx = \pi\Big[2x - \tan x\Big]_{-\pi/4}^{\pi/4}$$

$$= \pi\left[\left(\frac{\pi}{2}-1\right)-\left(-\frac{\pi}{2}+1\right)\right]$$

$$= \pi^2 - 2\pi.$$

20.



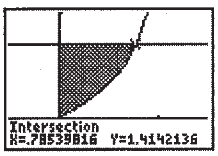
[–1, 5] by [–3, 1]

The curves intersect where $-\sqrt{x}=-2$, which is where $x=4$. Use washer cross sections: a washer has inner radius $r=\sqrt{x}$, outer radius $R=2$, and area

$A(x)=\pi(R^2-r^2)=\pi(4-x)$.

The volume is $\int_0^4 \pi(4-x)\,dx = \pi\left[4x-\frac{1}{2}x^2\right]_0^4 = 8\pi$

21.



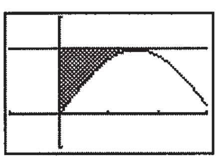
[–0.5, 1.5] by [–0.5, 2]

The curves intersect at $x=\frac{\pi}{4}\approx 0.7854$. A cross section has radius $r=\sqrt{2}-\sec x\tan x$ and area

$A(x)=\pi r^2=\pi(\sqrt{2}-\sec x\tan x)^2$. Use NINT to find

$\int_0^{0.7854} \pi(\sqrt{2}-\sec x\tan x)^2\,dx \approx 2.301.$

22.



[–1, 3] by [–1, 3]

The curve and horizontal line intersect at $x=\frac{\pi}{2}$. A cross section has radius 2 – 2 sin x and area

$A(x)=\pi r^2 = 4\pi(1-\sin x)^2 = 4\pi(1-2\sin x+\sin^2 x)$.

The volume is

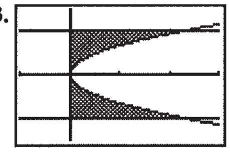
$$\int_0^{\pi/2} 4\pi(1-2\sin x+\sin^2 x)\,dx$$

$$= \int_0^{\pi/2} 4\pi\left(1-2\sin x+\frac{1-\cos 2x}{2}\right)dx$$

$$= 4\pi\left[\frac{3}{2}x+2\cos x-\frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$= 4\pi\left(\frac{3\pi}{4}-2\right)=\pi(3\pi-8)$$

23.



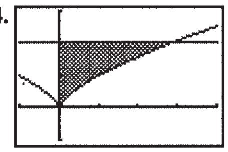
[–1, 3] by [–1.5, 1.5]

A cross section has radius $r=\sqrt{5}y^2$ and area

$A(y)=\pi r^2 = 5\pi y^4$.

The volume is $\int_{-1}^{1} 5\pi y^4\,dy = \pi\Big[y^5\Big]_{-1}^{1} = 2\pi.$

24.



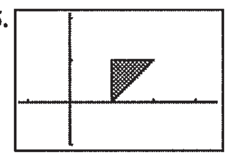
[–1, 4] by [–1, 3]

A cross section has radius $r=y^{3/2}$ and area

$A(y)=\pi r^2=\pi y^3$. The volume is

$\int_0^2 \pi y^3\,dy = \pi\left[\frac{1}{4}y^4\right]_0^2 = 4\pi.$

25.

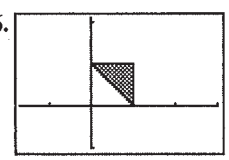


[–1.2, 3.5] by [–1, 2.1]

Use washer cross sections. A washer has inner radius $r=1$. Outer radius $R=y+1$, and area

$A(y)=\pi(R^2-r^2)=\pi\left[(y+1)^2-1\right]=\pi(y^2+2y)$. The volume is $\int_0^1 \pi(y^2+2y)\,dy = \pi\left[\frac{1}{3}y^3+y^2\right]_0^1 = \frac{4}{3}\pi.$

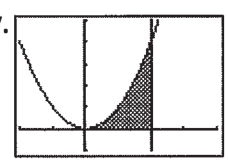
26.



[–1.7, 3] by [–1, 2.1]

Use cylindrical shells: a shell has radius x and height x. The volume is $\int_0^1 2\pi(x)(x)\,dx = 2\pi\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3}\pi.$

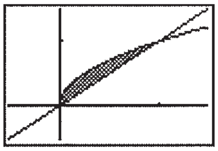
27.



[–2, 4] by [–1, 5]

Use cylindrical shells: A shell has radius x and height x^2.

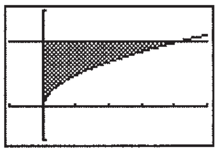
The volume is $\int_0^2 2\pi(x)(x^2)\,dx = 2\pi\left[\frac{1}{4}x^4\right]_0^2 = 8\pi.$

28. 

[−0.5, 1.5] by [−0.5, 1.5]

The curves intersect at $x = 0$ and $x = 1$. Use cylindrical shells: a shell has radius x and height $\sqrt{x} - x$. The volume is

$$\int_0^1 2\pi(x)(\sqrt{x} - x)\,dx = 2\pi\left[\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^1 = \frac{2\pi}{15}.$$

29. 

[−1, 5] by [−1, 3]

The curved and horizontal line intersect at (4, 2).

(a) Use washer cross sections: a washer has inner radius $r = \sqrt{x}$, outer radius $R = 2$, and area $A(x) = \pi(R^2 - r^2) = \pi(4 - x)$. The volume is

$$\int_0^4 \pi(4 - x)\,dx = \pi\left[4x - \frac{1}{2}x^2\right]_0^4 = 8\pi$$

(b) A cross section has radius $r = y^2$ and area $A(y) = \pi r^2 = \pi y^4$.

The volume is $\displaystyle\int_0^2 \pi y^4\,dy = \pi\left[\frac{1}{5}y^5\right]_0^2 = \frac{32\pi}{5}.$

(c) A cross section has radius $r = 2 - \sqrt{x}$ and area $A(x) = \pi r^2 = \pi(2 - \sqrt{x})^2 = \pi(4 - 4\sqrt{x} + x)$.

The volume is

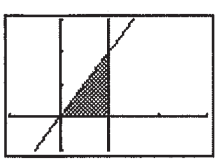
$$\int_0^4 \pi(4 - 4\sqrt{x} + x)\,dx = \pi\left[4x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^4 = \frac{8\pi}{3}.$$

(d) Use washer cross sections: a washer has inner radius $r = 4 - y^2$, outer radius $R = 4$, and area

$A(y) = \pi(R^2 - r^2) = \pi\left[16 - (4 - y^2)^2\right] = \pi(8y^2 - y^4).$

The volume is

$$\int_0^2 \pi(8y^2 - y^4)\,dy = \pi\left[\frac{8}{3}y^3 - \frac{1}{5}y^5\right]_0^2 = \frac{224\pi}{15}$$

30. 

[−1, 3] by [−1, 3]

The slanted and vertical lines intersect at (1, 2)

(a) The solid is a right circular cone of radius 1 and height 2. The volume is

$$\frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi 1^2)2 = \frac{2}{3}\pi.$$

Using integration: A cross section has radius $1 - \frac{y}{2}$ and area $A(y) = \pi r^2 = \pi\left(1 - \frac{y}{2}\right)^2$. The volume is

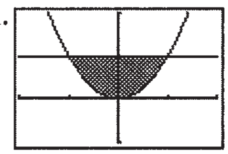
$$V = \int_0^2 \pi\left(1 - \frac{y}{2}\right)^2 dy = \pi\int_0^2\left(1 - y + \frac{1}{4}y^2\right)dy$$

$$= \pi\left[y - \frac{1}{2}y^2 + \frac{1}{12}y^3\right]_0^2 = \frac{2}{3}\pi.$$

(b) Use cylindrical shells: shell has radius $2 - x$ and height $2x$. The volume is

$$\int_0^1 2\pi(2 - x)(2x)\,dx = 4\pi\int_0^1 (2x - x^2)\,dx$$

$$= 4\pi\left[x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{8\pi}{3}.$$

31. 

[−2, 2] by [−1, 2]

The curves intersect at $(\pm 1, 1)$.

(a) A cross section has radius $r = 1 - x^2$ and area $A(x) = \pi r^2 = \pi(1 - x^2)^2 = \pi(1 - 2x^2 + x^4)$.

The volume is

$$\int_{-1}^1 \pi(1 - 2x^2 + x^4)\,dx = \pi\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^1 = \frac{16\pi}{15}.$$

(b) Use cylindrical shells: a shell has radius $2 - y$ and height $2\sqrt{y}$. The volume is

$$\int_0^1 2\pi(2 - y)(2\sqrt{y})\,dy = 4\pi\int_0^1 (2\sqrt{y} - y^{3/2})\,dy$$

$$= 4\pi\left[\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 = \frac{56\pi}{15}.$$

(c) Use cylindrical shells: a shell has radius $y + 1$ and height $2\sqrt{y}$. The volume is

$$\int_0^1 2\pi(y + 1)(2\sqrt{y})\,dy = 4\pi\int_0^1 (y^{3/2} + \sqrt{y})\,dy$$

$$= 4\pi\left[\frac{2}{5}y^{5/2} + \frac{2}{3}y^{3/2}\right]_0^1 = \frac{64\pi}{15}.$$

32. (a) A cross section has radius $r = h\left(1-\frac{x}{b}\right)$ and area

$A(x) = \pi r^2 = \pi h^2\left(1-\frac{x}{b}\right)^2$. The volume is

$$\int_0^b \pi h^2\left(1-\frac{x}{b}\right)^2 dx = \pi h^2\left[-\frac{b}{3}\left(1-\frac{x}{b}\right)^3\right]_0^b = \frac{\pi}{3}bh^2.$$

(b) Use cylindrical shells: a shell has radius x and height $h\left(1-\frac{x}{b}\right)$. The volume is

$$\int_0^b 2\pi(x)h\left(1-\frac{x}{b}\right)dx = 2\pi h\int_0^b\left(x-\frac{x^2}{b}\right)dx$$

$$=2\pi h\left[\frac{1}{2}x^2-\frac{x^3}{3b}\right]_0^b = \frac{\pi}{3}b^2h.$$

33. A shell has height $12(y^2-y^3)$.

(a) A shell has radius y. The volume is

$$\int_0^1 2\pi(y)12(y^2-y^3)\,dy = 24\pi\int_0^1(y^3-y^4)\,dy$$

$$=24\pi\left[\frac{1}{4}y^4-\frac{1}{5}y^5\right]_0^1 = \frac{6\pi}{5}.$$

(b) A shell has radius $1-y$. The volume is

$$\int_0^1 2\pi(1-y)12(y^2-y^3)\,dy$$

$$=24\pi\int_0^1(y^4-2y^3+y^2)\,dy$$

$$=24\pi\left[\frac{1}{5}y^5-\frac{1}{2}y^4+\frac{1}{3}y^3\right]_0^1 = \frac{4\pi}{5}.$$

(c) A shell has radius $\frac{8}{5}-y$. The volume is

$$\int_0^1 2\pi\left(\frac{8}{5}-y\right)12(y^2-y^3)\,dy$$

$$=24\pi\int_0^1\left(y^4-\frac{13}{5}y^3+\frac{8}{5}y^2\right)dy$$

$$=24\pi\left[\frac{1}{5}y^5-\frac{13}{20}y^4+\frac{8}{15}y^3\right]_0^1 = 2\pi.$$

(d) A shell has radius $y+\frac{2}{5}$. The volume is

$$\int_0^1 2\pi\left(y+\frac{2}{5}\right)12\left(y^2-y^3\right)dy$$

$$=24\pi\int_0^1\left(-y^4+\frac{3}{5}y^3+\frac{2}{5}y^2\right)dx$$

$$=24\pi\left[-\frac{1}{5}y^5+\frac{3}{20}y^4+\frac{2}{15}y^3\right]_0^1 = 2\pi.$$

34. A shell has height $\frac{y^2}{2}-\left(\frac{y^4}{4}-\frac{y^2}{2}\right)=y^2-\frac{y^4}{4}$.

(a) A shell has radius y. The volume is

$$\int_0^2 2\pi(y)\left(y^2-\frac{y^4}{4}\right)dy = 2\pi\left[\frac{1}{4}y^4-\frac{1}{24}y^6\right]_0^2 = \frac{8\pi}{3}.$$

(b) A shell has radius $2-y$. The volume is

$$\int_0^2 2\pi(2-y)\left(y^2-\frac{y^4}{4}\right)dy$$

$$=2\pi\int_0^2\left(\frac{y^5}{4}-\frac{y^4}{2}-y^3+2y^2\right)dy$$

$$=2\pi\left[\frac{1}{24}y^6-\frac{1}{10}y^5-\frac{1}{4}y^4+\frac{2}{3}y^3\right]_0^2 = \frac{8\pi}{5}.$$

(c) A shell has radius $5-y$. The volume is

$$\int_0^2 2\pi(5-y)\left(y^2-\frac{y^4}{4}\right)dy$$

$$=2\pi\int_0^2\left(\frac{y^5}{4}-\frac{5y^4}{4}-y^3+5y^2\right)dy$$

$$=2\pi\left[\frac{1}{24}y^6-\frac{1}{4}y^5-\frac{1}{4}y^4+\frac{5}{3}y^3\right]_0^2 = 8\pi.$$

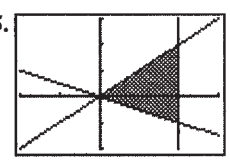
(d) A shell has radius $y+\frac{5}{8}$. The volume is

$$\int_0^2 2\pi\left(y+\frac{5}{8}\right)\left(y^2-\frac{y^4}{4}\right)dy$$

$$=2\pi\int_0^2\left(-\frac{y^5}{4}-\frac{5y^4}{32}+y^3+\frac{5y^2}{8}\right)dy$$

$$=2\pi\left[-\frac{1}{24}y^6-\frac{1}{32}y^5+\frac{1}{4}y^4+\frac{5}{24}y^3\right]_0^2 = 4\pi.$$

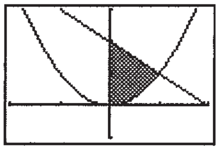
35.



[–2, 3] by [–2, 3]

A shell has radius x and height $x-\left(-\frac{x}{2}\right)=\frac{3}{2}x$.

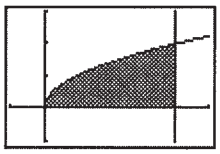
The volume is $\int_0^2 2\pi(x)\left(\frac{3}{2}x\right)dx = \pi\left[x^3\right]_0^2 = 8\pi$.

36. 

[–2, 2] by [–1, 3]

$x^2 = 2 - x$ at $x = 1$. A shell has radius x and height $2 - x - x^2$. The volume is

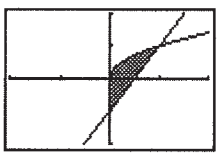
$$\int_0^1 2\pi(x)(2-x-x^2)\,dx = 2\pi\left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{5\pi}{6}.$$

37. 

[–1, 5] by [–1, 3]

A shell has radius x and height $\sqrt{x}$. The volume is

$$\int_0^4 2\pi(x)(\sqrt{x})\,dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}.$$

38. 

[–2, 2] by [–2, 2]

The functions intersect where $2x - 1 = \sqrt{x}$, i.e., at $x = 1$. A shell has radius x and height $\sqrt{x} - (2x-1) = \sqrt{x} - 2x + 1$. The volume is

$$\int_0^1 2\pi(x)(\sqrt{x} - 2x + 1)\,dx = 2\pi\int_0^1 (x^{3/2} - 2x^2 + x)\,dx$$

$$= 2\pi\left[\frac{2}{5}x^{5/2} - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1$$

$$= \frac{7\pi}{15}.$$

39. A cross section has width $w = 2\sqrt{\sin x}$.

(a) $A(x) = \frac{\sqrt{3}}{4}w^2 = \sqrt{3}\sin x$, and

$$V = \int_0^\pi \sqrt{3}\sin x\,dx$$
$$= \sqrt{3}\int_0^\pi \sin x\,dx$$
$$= \sqrt{3}\left[-\cos x\right]_0^\pi$$
$$= 2\sqrt{3}.$$

(b) $A(x) = s^2 = w^2 = 4\sin x$, and

$$V = \int_0^\pi 4\sin x\,dx = 4\int_0^\pi \sin x\,dx = 4\left[-\cos x\right]_0^\pi = 8.$$

40. A cross section has width $w = \sec x - \tan x$.

(a) $A(x) = \pi r^2 = \pi\left(\frac{w}{2}\right)^2 = \frac{\pi}{4}(\sec x - \tan x)^2$, and

$$V = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}(\sec x - \tan x)^2\,dx$$
$$= \frac{\pi}{4}\int_{-\pi/3}^{\pi/3}(\sec^2 x - 2\sec x\tan x + \tan^2 x)\,dx$$
$$= \frac{\pi}{4}\left[\tan x - 2\sec x + \tan x - x\right]_{-\pi/3}^{\pi/3}$$
$$= \frac{\pi}{2}\left[\tan x - \sec x - \frac{1}{2}x\right]_{-\pi/3}^{\pi/3}$$
$$= \frac{\pi}{2}\left[\left(\sqrt{3} - 2 - \frac{\pi}{6}\right) - \left(-\sqrt{3} - 2 + \frac{\pi}{6}\right)\right]$$
$$= \pi\sqrt{3} - \frac{\pi^2}{6}.$$

(b) $A(x) = s^2 = w^2 = (\sec x - \tan x)^2$, and

$V = \int_{-\pi/3}^{\pi/3}(\sec x - \tan x)^2\,dx$, which by same method as in part (a) equals $4\sqrt{3} - \frac{2}{3}\pi$.

41. A cross section has width $w = \sqrt{5}y^2$ and area $\pi r^2 = \pi\left(\frac{w}{2}\right)^2 = \frac{5\pi}{4}y^4$. The volume is

$$\int_0^2 \frac{5\pi}{4}y^4\,dy = \frac{\pi}{4}\left[y^5\right]_0^2 = 8\pi.$$

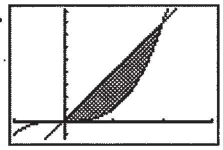
42. A cross section has width $w = 2\sqrt{1-y^2}$ and area $\frac{1}{2}s^2 = \frac{1}{2}w^2 = 2(1-y^2)$. This volume is

$$\int_{-1}^1 2(1-y^2)\,dy = 2\left[y - \frac{1}{3}y^3\right]_{-1}^1 = \frac{8}{3}.$$

43. Since the diameter of the circular base of the solid extends from $y = \frac{12}{2} = 6$ to $y = 12$, for a diameter of 6 and a radius of 3, the solid has the same cross sections as the right circular cone. The volumes are equal by Cavalieri's Theorem.

44. **(a)** The volume is the same as if the square had moved without twisting: $V = Ah = s^2h$.

(b) Still s^2h: the lateral distribution of the square cross sections doesn't affect the volume. That's Cavalieri's Volume Theorem.

45. 

$[-1, 3]$ by $[-1.4, 9.1]$

The functions intersect at (2, 8).

(a) Use washer cross sections: a washer has inner radius $r = x^3$, outer radius $R = 4x$, and area $A(x) = \pi(R^2 - r^2) = \pi(16x^2 - x^6)$. The volume is

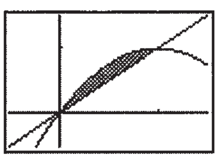
$$\int_0^2 \pi(16x^2 - x^6)\,dx = \pi\left[\frac{16}{3}x^3 - \frac{1}{7}x^7\right]_0^2 = \frac{512\pi}{21}.$$

(b) Use cylindrical shells: a shell has a radius $8 - y$ and height $y^{1/3} - \frac{y}{4}$. The volume is

$$\int_0^8 2\pi(8-y)\left(y^{1/3} - \frac{y}{4}\right)dy$$

$$= 2\pi\int_0^8\left(8y^{1/3} - 2y - y^{4/3} + \frac{y^2}{4}\right)dy$$

$$= 2\pi\left[6y^{4/3} - y^2 - \frac{3}{7}y^{7/3} + \frac{1}{12}y^3\right]_0^8 = \frac{832\pi}{21}.$$

46. 

$[-0.5, 1.5]$ by $[-0.5, 1.5]$

The functions intersect at (0, 0) and (1, 1).

(a) Use cylindrical shells: A shell has radius x and height $2x - x^2 - x = x - x^2$. The volume is

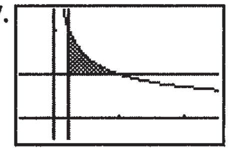
$$\int_0^1 2\pi(x)(x - x^2)\,dx = 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{\pi}{6}.$$

(b) Use cylindrical shells: a shell has radius $1 - x$ and height $2x - x^2 - x = x - x^2$. The volume is

$$\int_0^1 2\pi(1-x)(x - x^2)dx = 2\pi\int_0^1 (x^3 - 2x^2 + x)\,dx$$

$$= 2\pi\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1$$

$$= \frac{\pi}{6}.$$

47. 

$[-0.5, 2.5]$ by $[-0.5, 2.5]$

The intersection points are $\left(\frac{1}{4}, 1\right), \left(\frac{1}{4}, 2\right)$, and (1, 1).

(a) A washer has inner radius $r = \frac{1}{4}$, outer radius $R = \frac{1}{y^2}$, and area $\pi(R^2 - r^2) = \pi\left(\frac{1}{y^4} - \frac{1}{16}\right)$. The volume is

$$\int_1^2 \pi\left(\frac{1}{y^4} - \frac{1}{16}\right)dy = \pi\left[-\frac{1}{3y^3} - \frac{1}{16}y\right]_1^2 = \frac{11\pi}{48}.$$

(b) A shell has radius x and height $\frac{1}{\sqrt{x}} - 1$. The volume is

$$\int_{1/4}^1 2\pi(x)\left(\frac{1}{\sqrt{x}} - 1\right)dx = 2\pi\left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2\right]_{1/4}^1 = \frac{11\pi}{48}.$$

48. (a) For $0 < x \le \pi$, $x\,f(x) = \frac{x(\sin x)}{x} = \sin x$.
For $x = 0$, $x\,f(x) = 0 \bullet 1 = \sin 0 = \sin x$. so $x\,f(x) = \sin x$ for $0 \le x \le \pi$.

(b) Use cylindrical shells: a shell has radius x and height y. The volume is $\int_0^\pi 2\pi xy\,dx$, which from part (a) is $\int_0^\pi 2\pi\sin x\,dx = 2\pi\left[-\cos x\right]_0^\pi = 4\pi$.

49. (a) A cross section has radius $r = \frac{x}{12}\sqrt{36 - x^2}$ and area $A(x) = \pi r^2 = \frac{\pi}{144}(36x^2 - x^4)$. The volume is

$$\int_0^6 \frac{\pi}{144}(36x^2 - x^4)\,dx = \frac{\pi}{144}\left[12x^3 - \frac{1}{5}x^5\right]_0^6$$

$$= \frac{36\pi}{5}\text{cm}^3.$$

(b) $\left(\frac{36\pi}{5}\text{cm}^3\right)(8.5\,\text{g}/\text{cm}^3) \approx 192.3\,\text{g}.$

50. (a) A cross section has radius $r = \sqrt{2y}$ and area $\pi r^2 = 2\pi y$. The volume is $\int_0^5 2\pi y\,dy = \pi\left[y^2\right]_0^5 = 25\pi$.

50. Continued

(b) $V(h) = \int A(h)\,dh$, so $\frac{dV}{dh} = A(h)$.

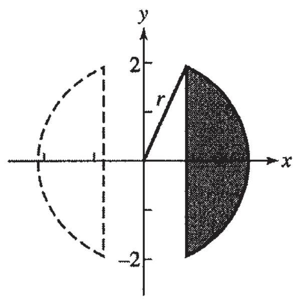
$\therefore \frac{dV}{dt} = \frac{dV}{dh} \bullet \frac{dh}{dt} = A(h) \bullet \frac{dh}{dt}$,

so $\frac{dh}{dt} = \frac{1}{A(h)} \bullet \frac{dV}{dt}$

For $h = 4$, the area is $2\pi(4) = 8\pi$,

so $\frac{dh}{dt} = \frac{1}{8\pi} \bullet 3\frac{\text{units}^3}{\text{sec}} = \frac{3}{8\pi} \bullet \frac{\text{units}^3}{\text{sec}}$.

51. (a)



The remaining solid is that swept out by the shaded region in revolution. Use cylindrical shells: a shell has radius x and height $2\sqrt{r^2 - x^2}$. The volume is

$$\int_{\sqrt{r^2-2^2}}^{r} 2\pi(x)\left(2\sqrt{r^2 - x^2}\right)dx$$

$$= 2\pi\left[-\frac{2}{3}(r^2 - x^2)^{3/2}\right]_{\sqrt{r^2-4}}^{r}$$

$$= -\frac{4}{3}\pi(-8) = \frac{32\pi}{3}.$$

(b) The answer is independent of r.

52. Partition the appropriate interval in the axis of revolution and measure the radius $r(x)$ of the shadow region at these points. Then use an approximation such as the trapezoidal rule to estimate the integral $\int_a^b \pi r^2(x)\,dx$.

53. Solve $ax - x^2 = 0$: This is true at $x = a$. For revolution about the x-axis, a cross section has radius $r = ax - x^2$ and area $A(x) = \pi r^2 = \pi(ax - x^2)^2 = \pi(a^2x^2 - 2ax^3 + x^4)$. The volume is

$$\int_0^a \pi(a^2x^2 - 2ax^3 + x^4)\,dx = \pi\left[\frac{1}{3}a^2x^3 - \frac{1}{2}ax^4 + \frac{1}{5}x^5\right]_0^a$$

$$= \frac{1}{30}\pi a^5.$$

For revolution about the y-axis, a cylindrical shell has radius x and height $ax - x^2$. The volume is

$$\int_0^a 2\pi(x)(ax - x^2)\,dx = 2\pi\left[\frac{1}{3}ax^3 - \frac{1}{4}x^4\right]_0^a = \frac{1}{6}\pi a^4.$$

Setting the two volumes equal,

$\frac{1}{30}\pi a^5 = \frac{1}{6}\pi a^4$ yields $\frac{1}{30}a = \frac{1}{6}$, so $a = 5$.

54. The slant height Δs of a tiny horizontal slice can be written as $\Delta s = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + (g'(y))^2}\,\Delta y$. So the surface area is approximated by the Riemann sum

$\sum_{k=1}^{n} 2\pi g(y_k)\sqrt{1 + (g'(y))^2}\,\Delta y$. The limit of that is the integral.

55. $g'(y) = \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$, and

$$\int_0^2 2\pi\sqrt{y}\sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\,dy = \int_0^2 \pi\sqrt{4y + 1}\,dy$$

$$= \left[\frac{\pi}{6}(4y + 1)^{3/2}\right]_0^2$$

$$= \frac{13\pi}{3} \approx 13.614$$

56. $g'(y) = \frac{dx}{dy} = y^2$, and

$$\int_0^1 2\pi\left(\frac{y^3}{3}\right)\sqrt{1 + (y^2)^2}\,dy = \frac{2}{3}\pi\left[\frac{1}{6}(1 + y^4)^{3/2}\right]_0^1$$

$$= \frac{\pi}{9}(2\sqrt{2} - 1) \approx 0.638.$$

57. $g'(y) = \frac{dx}{dy} = \frac{1}{2}y^{-1/2}$, and

$$\int_1^3 2\pi\left[y^{1/2} - \left(\frac{1}{3}\right)^{3/2}\right]\sqrt{1 + \left(\frac{1}{2}y^{-1/2}\right)^2}\,dy$$

$$= 2\pi\int_1^3\left[y^{1/2} - \left(\frac{1}{3}\right)^{3/2}\right]\sqrt{1 + \frac{1}{4y}}\,dy.$$

Using NINT, this evaluates to ≈ 16.110

58. $g'(y) = \frac{dx}{dy} = \frac{1}{\sqrt{2y - 1}}$, and

$$\int_{5/8}^{1} 2\pi\sqrt{2y - 1}\sqrt{1 + \left(\frac{1}{2y - 1}\right)^2}\,dy$$

$$= 2\pi\int_{5/8}^{1}\sqrt{2y}\,dy$$

$$= 2\sqrt{2}\pi\left[\frac{2}{3}y^{3/2}\right]_{5/8}^{1}$$

$$= \frac{4\sqrt{2}}{3}\pi\left(1 - \frac{5}{16}\sqrt{\frac{5}{2}}\right) \approx 2.997.$$

59. $f'(x) = \frac{dy}{dx} = 2x$, and

$\int_0^2 2\pi x^2\sqrt{1 + (2x)^2}\,dx = \int_0^2 2\pi x^2\sqrt{1 + 4x^2}\,dx$ evaluates, using NINT, to ≈ 53.226.

60. $f'(x)=\frac{dy}{dx}=3-2x$, and

$\int_0^3 2\pi(3x-x^2)\sqrt{1+(3x-2x)^2}\,dx$ evaluates, using NINT, to ≈ 44.877.

61. $f'(x)=\frac{dy}{dx}=\frac{1-x}{\sqrt{2x-x^2}}$, and

$$\int_{0.5}^{1.5} 2\pi\sqrt{2x-x^2}\sqrt{1+\left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2}\,dx=2\pi\int_{0.5}^{1.5}1\,dx$$
$$=2\pi\left[x\right]_{0.5}^{1.5}$$
$$=2\pi\approx 6.283$$

62. $f'(x)=\frac{dy}{dx}=\frac{1}{2\sqrt{x+1}}$, and

$$\int_1^5 2\pi\sqrt{x+1}\sqrt{1+\left(\frac{1}{2\sqrt{x+1}}\right)^2}\,dx$$
$$=2\pi\int_1^5\sqrt{x+\frac{5}{4}}\,dx$$
$$=2\pi\left[\frac{2}{3}\left(x+\frac{5}{4}\right)^{3/2}\right]_1^5$$
$$=\frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{3/2}-\left(\frac{9}{4}\right)^{3/2}\right]=\frac{49\pi}{3}\approx 51.313$$

63. True. by definition

64. False. The volume is given by $\int_0^2 \pi y^4\,dy$.

65. A. $V=\int_1^e (\ln(x))^2\,dx=0.718$

66. E. $V=\int_0^4 (\pi(8-x^{3/2})^2)\,dx=361.9$

67. B. $V=\int_0^{16}\pi\left(4^2-\left(\sqrt{y}\right)^2\right)dy=128\pi$

68. D.

69. A cross section has radius $r=|c-\sin x|$ and area

$A(x)=\pi r^2=\pi(c-\sin x)^2=\pi(c^2-2c\sin x+\sin^2 x)$.

The volume is

$$\int_0^{\pi}\pi(c^2-2c\sin x+\sin^2 x)\,dx$$
$$=\pi\left[c^2x-2c\cos x+\frac{1}{2}x-\frac{1}{4}\sin 2x\right]_0^{\pi}$$
$$=\pi\left[\left(c^2\pi-2c+\frac{1}{2}\pi\right)-2c\right]$$
$$=\pi^2c^2-4\pi c+\frac{\pi^2}{2}.$$

(a) Solve

$$\frac{d}{dc}\left[\pi^2c^2-4\pi c+\frac{\pi^2}{2}\right]=0$$
$$2\pi^2c-4\pi=0$$
$$\pi c-2=0$$
$$c=\frac{2}{\pi}$$

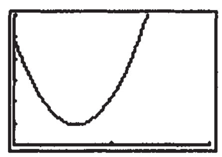
This value of c gives a minimum for V because

$\frac{d^2V}{dc^2}=2\pi^2>0$.

Then the volume is $\pi^2\left(\frac{2}{\pi}\right)^2-4\pi\left(\frac{2}{\pi}\right)+\frac{\pi^2}{2}=\frac{\pi^2}{2}-4$

(b) Since the derivative with respect to c is not zero anywhere else besides $c=\frac{2}{\pi}$, the maximum must occur at $c=0$ or $c=1$. The volume for $c=0$ is $\frac{\pi^2}{2}\approx 4.935$, and for $c=1$ it is $\frac{\pi(3\pi-8)}{2}\approx 2.238$. $c=0$ maximizes the volume.

(c)



[0, 2] by [0, 6]

The volume gets large without limit. This makes sense, since the curve is sweeping out space in an ever-increasing radius.

70. (a) Using $d=\frac{C}{\pi}$, and $A=\pi\left(\frac{d}{2}\right)^2=\frac{C^2}{4\pi}$ yields the following areas (in square inches, rounded to the nearest tenth): 2.3, 1.6, 1.5, 2.1, 3.2, 4.8, 7.0, 9.3, 10.7, 10.7, 9.3, 6.4, 3.2.

(b) If $C(y)$ is the circumference as a function of y, then the area of a cross section is

$A(y)=\pi\left(\frac{C(y)/\pi}{2}\right)^2=\frac{(C(y))^2}{4\pi}$, and the volume is

$\frac{1}{4\pi}\int_0^6 (C(y))^2\,dy$.

(c) $\frac{1}{4\pi}\int_0^6 A(y)\,dy=\frac{1}{4\pi}\int_0^6 (C(y))^2\,dy$

$$\approx\frac{1}{4\pi}\left(\frac{6-0}{24}\right)[5.4^2+2(4.5^2+4.4^2$$
$$+5.1^2+6.3^2+7.8^2+9.4^2+10.8^2+11.6^2$$
$$+11.6^2+10.8^2+9.0^2)+6.3^2]\approx 34.7\text{ in.}^3$$

71. Hemisphere cross sectional area: $\pi\left(\sqrt{R^2-h^2}\right)^2 = A_1$.

Right circular cylinder with cone removed cross sectional area: $\pi R^2 - \pi h^2 = A_2$

Since $A_1 = A_2$, the two volumes are equal by Cavalieri's theorem. Thus volume of hemisphere
= volume of cylinder – volume of cone

$= \pi R^3 - \frac{1}{3}\pi R^3 = \frac{2}{3}\pi R^3$.

72. Use washer cross sections: a washer has inner radius $r = b - \sqrt{a^2-y^2}$, outer radius $R = b + \sqrt{a^2+y^2}$, and area

$\pi(R^2 - r^2)$

$= \pi\left[\left(b+\sqrt{a^2-y^2}\right)^2 - \left(b-\sqrt{a^2-y^2}\right)^2\right]$

$= 4\pi b\sqrt{a^2-y^2}$. The volume is

$$\int_{-a}^{a} 4\pi b\sqrt{a^2-y^2}\,dy = 4\pi b\int_{-a}^{a}\sqrt{a^2-y^2}\,dy$$
$$= 4\pi b\left(\frac{\pi a^2}{2}\right)$$
$$= 2\pi^2 a^2 b$$

73. **(a)** Put the bottom of the bowl at $(0, -a)$. The area of a horizontal cross section is $\pi\left(\sqrt{a^2-y^2}\right)^2 = \pi(a^2-y^2)$.

The volume for height h is

$$\int_{-a}^{h-a} \pi(a^2-y^2)\,dy = \pi\left[a^2y - \frac{1}{3}y^3\right]_{-a}^{h-a} = \frac{\pi h^2(3a-h)}{3}.$$

(b) For $h = 4, y = -1$ and the area of a cross section is $\pi(5^2 - 1^2) = 24\pi$. The rate of rise is

$$\frac{dh}{dt} = \frac{1}{A}\frac{dV}{dt} = \frac{1}{24\pi}(0.2) = \frac{1}{120\pi} \text{ m/sec.}$$

74. **(a)** A cross section has radius $r = \sqrt{a^2-x^2}$ and area $A(x) = \pi r^2 = \pi\left(\sqrt{a^2-x^2}\right)^2 = \pi(a^2-x^2)$. The volume is

$$\int_{-a}^{a} \pi(a^2-x^2)\,dx = \pi\left[a^2x - \frac{1}{3}x^3\right]_{-a}^{a}$$
$$= \pi\left[\left(a^3 - \frac{1}{3}a^3\right) - \left(-a^3 + \frac{1}{3}a^3\right)\right]$$
$$= \frac{4}{3}\pi a^3.$$

(b) A cross section has radius $x = r\left(1 - \frac{y}{h}\right)$ and area

$A(y) = \pi x^2 = \pi r^2\left(1-\frac{y}{h}\right)^2 = \pi r^2\left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right)$. The volume is

$$\int_0^h \pi r^2\left(1 - \frac{2y}{h} + \frac{y^2}{h}\right)dy = \pi r^2\left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_0^h$$
$$= \frac{1}{3}\pi r^2 h.$$

Quick Quiz Sections 7.1–7.3

1. C. $\int_0^1 (\sin^{-1}(x))^2\,dx = 0.467$

2. A.

3. D.

4. **(a)** The two graphs intersect where $\sqrt{x} = e^{-x}$, which a calculator shows to be $x = 0.42630275$.
Store this value as A.
The area of R is $\int_0^A (e^{-x} - \sqrt{x})\,dx = 0.162$.

(b) Volume $= \int_0^A \pi\left(\left(e^{-x}+1\right)^2 - \left(\sqrt{x}+1\right)^2\right)dx = 1.631$.

(c) Volume $= \int_0^A \frac{1}{2}\pi\left(\frac{e^{-x}-\sqrt{x}}{2}\right)^2 dx = 0.035$.

Section 7.4 Lengths of Curves (pp. 412–418)

Quick Review 7.4

1. $\sqrt{1+2x+x^2} = \sqrt{(1+x)^2}$, which, since $x \ge -1$, equals $1 + x$ or $x + 1$.

2. $\sqrt{1-x+\frac{x^2}{4}} = \sqrt{\left(1-\frac{x}{2}\right)^2}$, which, since $x \le 2$, equals $1 - \frac{x}{2}$ or $\frac{2-x}{2}$.

3. $\sqrt{1+(\tan x)^2} = \sqrt{(\sec x)^2}$, which since $0 \le x < \frac{\pi}{2}$, equals $\sec x$.

4. $\sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{16}x^2+\frac{1}{x^2}} = \frac{1}{4}\sqrt{\frac{(x^2+4)^2}{x^2}}$ which, since $x > 0$, equals $\frac{x^2+4}{4x}$.