Section 7.2 Areas in the Plane (pp. 390–398)

Exploration 1 A Family of Butterflies

1. For $k = 1$:

$$\int_0^{\pi} [(2-\sin x)-\sin x]\,dx = \int_0^{\pi}(2-2\sin x)\,dx$$
$$= [2x+2\cos x]_0^{\pi}$$
$$= 2\pi - 4$$

For $k = 2$:

$$\int_0^{\pi/2} [(4-2\sin 2x)-(2\sin 2x)]\,dx$$
$$= \int_0^{\pi/2}(4-4\sin 2x)\,dx$$
$$= [4x+2\cos 2x]_0^{\pi/2} = 2\pi - 4$$

2. It appears that the areas for $k \geq 3$ will continue to be $2\pi - 4$.

3. $A_k = \int_0^{\pi/k} [(2k - k\sin kx) - k\sin kx]\,dx$

$$= \int_0^{\pi/k} [(2k - 2k\sin kx)\,dx$$

If we make the substitution $u = kx$, then $du = k\,dx$ and the u-limits become 0 to π Thus,

$$A_k = \int_0^{\pi/k} [(2k - 2k\sin kx)\,dx$$
$$= \int_0^{\pi/k} [(2 - 2\sin kx)k\,dx$$
$$= \int_0^{\pi} (2-2\sin u)\,du.$$

4. $2\pi - 4$

5. Because the amplitudes of the sine curves are k, the kth butterfly stands $2k$ units tall. The vertical edges alone have lengths $(2k)$ that increase without bound, so the perimeters are tending to infinity.

Quick Review 7.2

1. $\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -[-1-1] = 2$

2. $\int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}(e^2-1) \approx 3.195$

3. $\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx = [\tan x]_{-\pi/4}^{\pi/4} = 1-(-1) = 2$

4. $\int_0^2 (4x - x^3)\,dx = \left[2x^2 - \frac{1}{4}x^4\right]_0^2 = (8-4)-0 = 4$

5. $\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2}$ (This is half the area of a circle of radius 3.)

6. Solve $x^2 - 4x = x + 6$.

$x^2 - 5x - 6 = 0$
$(x-6)(x+1) = 0$
$x = 6$ or $x = -1$
$y = 6+6 = 12$ or $y = -1+6 = 5$
$(6, 12)$ and $(-1, 5)$

7. Solve $e^x = x + 1$. From the graphs, it appears that e^x is always greater than or equal to $x + 1$, so that if they are ever equal, this is when $e^x - (x+1)$ is at a minimum.

$\frac{d}{dx}\left[e^x - (x+1)\right] = e^x - 1$ is zero when $e^x = 1$, i.e., when $x = 0$. Test: $e^0 = 0 + 1 = 1$. So the solution is (0, 1).

8. Inspection of the graphs shows two intersection points: (0, 0), and (π, 0). Check: $0^2 - \pi \cdot 0 = \sin 0 = 0$ and $\pi^2 - \pi^2 = \sin \pi = 0$.

9. Solve $\frac{2x}{x^2+1} = x^3$.

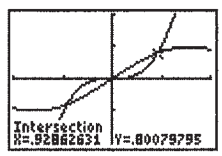
(0, 0) is a solution. Now divide by x.

$\frac{2}{x^2+1} = x^2$
$2 = x^4 + x^2$
$x^4 + x^2 - 2 = 0$
$x^2 = \frac{-1 \pm \sqrt{1+8}}{2} = -2$ or 1

Throw out the negative solution.

$x = \pm 1$
$y = x^3 = \pm 1$
(0, 0), (–1, –1) and (1, 1)

10. Use the intersect function on a graphing calculator:



$[-2, 2]$ by $[-2, 2]$

(–0.9286, –0.8008), (0, 0), and (0.9286, 0.8008)

Section 7.2 Exercises

1. $\int_0^{\pi} (1-\cos^2 x)\,dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}$

2. Use symmetry:

$$\int_{-\pi/3}^{\pi/3} \left(\frac{1}{2}\sec^2 t + 4\sin^2 t\right)dt = \int_0^{\pi/3}(\sec^2 + 8\sin^2 t)\,dt$$
$$= [\tan t + 4t - 2\sin 2t]_0^{\pi/3}$$
$$= \left(\sqrt{3} + \frac{4\pi}{3} - \sqrt{3}\right) - 0$$
$$= \frac{4\pi}{3}$$

3. $\int_0^1 (y^2 - y^3)\,dy = \left[\frac{1}{3}y^3 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{12}$

4. $\int_0^1 \left[(12y^2-12y^3)-(2y^2-2y)\right]dy$

$= \int_0^1 (-12y^3+10y^2+2y)\,dx$

$= \left[-3y^4+\frac{10}{3}y^3+y^2\right]_0^1$

$= -3+\frac{10}{3}+1=\frac{4}{3}$

5. Use the region's symmetry:

$2\int_0^2 [2x^2-(x^4-2x^2)]dx = 2\int_0^2 (-x^4+4x^2)dx$

$= 2\left[-\frac{1}{5}x^5+\frac{4}{3}x^3\right]_0^2$

$= 2\left[\left(-\frac{32}{5}+\frac{32}{3}\right)-0\right]=\frac{128}{15}$

6. Use the region's symmetry:

$$2\int_0^1 (x^2+2x^4)dx = 2\left[\frac{1}{3}x^3+\frac{2}{5}x^5\right]_0^1 = 2\left(\frac{1}{3}+\frac{2}{5}\right)=\frac{22}{15}$$

7. The functions intersect at $x\approx -1.4096$ and $x\approx 0.6367$.

$\int_{-1.4096}^{0.6367}[(1-x^2)-\sin x]dx \approx 1.670.$

8. The functions intersect at $x\approx \pm 1.152961$.

$\int_{-1.152961}^{1.152961}[\cos 2x-(x^2-2)]dx \approx 4.332.$

9. $\int_0^1\left(x-\frac{x^2}{4}\right)dx+\int_1^2\left(1-\frac{x^2}{4}\right)dx$

$= \left(\frac{x^2}{2}-\frac{x^3}{12}\right)\Bigg|_0^1 + \left(x-\frac{x^3}{12}\right)\Bigg|_1^2$

$= \left(\frac{1^2}{2}-\frac{1^3}{12}\right)-\left(\frac{0^2}{12}-\frac{0^3}{12}\right)+\left(2-\frac{2^3}{12}\right)-\left(1-\frac{1^3}{12}\right)$

$= 5/6$

10. $\int_0^1 x^2dx+\int_1^2(-x+2)dx$

$= \frac{x^3}{3}\Bigg|_0^1 + \left(-\frac{x^2}{2}+2x\right)\Bigg|_1^2$

$= \frac{1^3}{3}-0+\left(-\frac{2^2}{2}+2(2)\right)-\left(-\frac{1^2}{2}+2(1)\right)$

$= 5/6$

11. $\int_{-\sqrt{2}}^{\sqrt{2}}(3-y^2-(y^2-1))dy$

$= \int_{-\sqrt{2}}^{\sqrt{2}}(4-2y^2)dy = \left(4y-\frac{2y^3}{3}\right)\Bigg|_{-\sqrt{2}}^{\sqrt{2}}$

$= 4\sqrt{2}-\frac{2(\sqrt{2})^3}{3}-\left(4(-\sqrt{2})-\frac{2(-\sqrt{2})^3}{3}\right)$

$= \frac{16\sqrt{2}}{3}\approx 7.542$

12. $\int_{-3/2}^{2}\left(\frac{y}{2}-(y^2-3)\right)dy$

$= \left(\frac{y^2}{4}-\frac{y^3}{3}+3y\right)\Bigg|_{-3/2}^{2}$

$= \frac{(2)^2}{4}-\frac{(2)^3}{3}+3(2)-\left(\frac{\left(-\frac{3}{2}\right)^2}{4}-\frac{\left(-\frac{3}{2}\right)^3}{3}+3\left(-\frac{3}{2}\right)\right)$

$= 7\frac{7}{48}\approx 7.146$

13. $\int_{-2}^{0}\left((2x^3-x^2-5x)-(-x^2+3x)\right)dx$

$+\int_0^2\left(-x^2+3x-(2x^3-x^2-5x)\right)dx$

$= \int_{-2}^{0}(2x^3-8x)dx+\int_0^2(-2x^3+8x)dx$

$= \left(\frac{x^4}{2}-4x^2\right)\Bigg|_{-2}^{0}+\left(-\frac{x^4}{2}+4x^2\right)\Bigg|_0^2$

$= \left(0-\left(\frac{(-2)^4}{2}-4(-2)^2\right)\right)+\left(\left(-\frac{2^4}{2}+4(2)^2\right)-0\right)$

$= \left(0-\left(\frac{16}{2}-16\right)\right)+\left(\left(-\frac{16}{2}+16\right)-0\right)$

$= 16$

14. $\int_{-2}^{-1}\left(-x+2-\left(4-x^2\right)\right)dx+\int_{-1}^{2}\left(4-x^2-\left(-x+2\right)\right)dx$

$+\int_2^3(-x+2)-(4-x^2)dx$

$= \int_{-2}^{-1}(-x-2+x^2)dx+\int_{-1}^{2}(x+2-x^2)dx$

$+\int_2^3(-x-2+x^2)dx$

$= \left(-\frac{x^2}{2}-2x+\frac{x^3}{3}\right)\Bigg|_{-2}^{-1}+\left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)\Bigg|_{-1}^{2}$

$+\left(-\frac{x^2}{2}-2x+\frac{x^3}{3}\right)\Bigg|_2^3$

$= \left(-\frac{(-1)^2}{2}-2(-1)+\frac{(-1)^3}{3}-\left(-\frac{(-2)^2}{2}-2(-2)+\frac{(-2)^3}{3}\right)\right)$

$+\left(\frac{2^2}{2}+2(2)-\frac{2^3}{3}-\left(\frac{(-1)^2}{2}+2(-1)-\frac{(-1)^3}{3}\right)\right)$

$+\left(-\frac{3^2}{2}-2(3)+\frac{3^3}{3}-\left(-\frac{2^2}{2}-2(2)+\frac{2^3}{3}\right)\right)$

$= 8\frac{1}{6}$

15. Solve $x^2-2=2$: $x^2=4$, so the curves intersect at $x=\pm 2$.

$\int_{-2}^{2}\left[2-(x^2-2)\right]dx=\int_{-2}^{2}(4-x^2)dx$

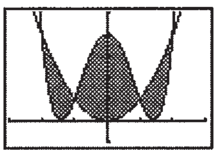
$= \left[4x-\frac{1}{3}x^3\right]_{-2}^{2} = \left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)=\frac{32}{3}=10\frac{2}{3}$

16. Solve $2x-x^2=-3$: $x^2-2x-3=(x-3)(x+1)=0$, so the curves intersect at $x=-1$ and $x=3$.

$$\int_{-1}^{3}\left(2x-x^2+3\right)dx=\left[x^2-\frac{1}{3}x^3+3x\right]_{-1}^{3}$$
$$=(9-9+9)-\left(1+\frac{1}{3}-3\right)$$
$$=\frac{32}{3}=10\frac{2}{3}$$

17. Solve $7-2x^2=x^2+4$: $x^2=1$, so the curves intersect at $x=\pm1$.

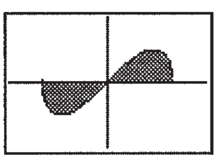
$$\int_{-1}^{1}\left[(7-2x^2)-(x^2+4)\right]dx=\int_{-1}^{1}(-3x^2+3)\,dx$$
$$=3\int_{-1}^{1}(1-x^2)\,dx$$
$$=3\left[x-\frac{1}{3}x^3\right]_{-1}^{1}$$
$$=3\left[\frac{2}{3}-\left(-\frac{2}{3}\right)\right]=4$$

18. 

[−3, 3] by [−1, 5]

The curves intersect at $x=\pm1$ and $x=\pm2$. Use the region's symmetry:

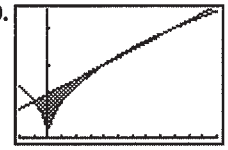
$$2\int_0^1\left[\left(x^4-4x^2+4\right)-x^2\right]dx+2\int_1^2\left[x^2-\left(x^4-4x^2+4\right)\right]dx$$
$$=2\int_0^1(x^4-5x^2+4)\,dx+2\int_1^2(-x^4+5x^2-4)\,dx$$
$$=2\left[\frac{1}{5}x^5-\frac{5}{3}x^3+4x\right]_0^1+2\left[-\frac{1}{5}x^5+\frac{5}{3}x^3-4x\right]_1^2$$
$$=2\left[\frac{1}{5}-\frac{5}{3}+4\right]+2\left[\left(-\frac{32}{5}+\frac{40}{3}-8\right)-\left(-\frac{1}{5}+\frac{5}{3}-4\right)\right]=8$$

19. 

$\left[-\frac{3}{2}a, \frac{3}{2}a\right]$ by $[-a^2, a^2]$

The curves intersect at $x=0$ and $x=\pm a$. Use the region's symmetry:

$$2\int_0^a x\sqrt{a^2-x^2}\,dx=2\left[-\frac{1}{3}(a^2-x^2)^{3/2}\right]_0^a$$
$$=2\left[0-\left(-\frac{1}{3}a^3\right)\right]$$
$$=\frac{2}{3}a^3$$

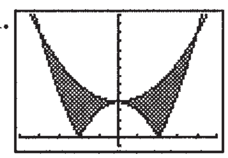
20. 

[−2, 12] by [0, 3.5]

The curves intersect at three points: $x=-1, x=4$ and $x=9$.

Because of the absolute value sign, break the integral up at $x=0$ also:

$$\int_{-1}^{0}\left(\frac{x+6}{5}-\sqrt{-x}\right)dx+\int_0^4\left(\frac{x+6}{5}-\sqrt{x}\right)dx+\int_4^9\left(\sqrt{x}-\frac{x+6}{5}\right)dx$$
$$=\left[\frac{\frac{1}{2}x^2+6x}{5}+\frac{2}{3}(-x)^{3/2}\right]_{-1}^{0}+\left[\frac{\frac{1}{2}x^2+6x}{5}-\frac{2}{3}x^{3/2}\right]_0^4+$$
$$\left[\frac{2}{3}x^{3/2}-\frac{\frac{1}{2}x^2+6x}{5}\right]_4^9$$
$$=\left[0-\left(-\frac{11}{10}+\frac{2}{3}\right)\right]+\left[\left(\frac{32}{5}-\frac{16}{3}\right)-0\right]+$$
$$\left[\left(18-\frac{189}{10}\right)-\left(\frac{16}{3}-\frac{32}{5}\right)\right]$$
$$=\frac{13}{30}+\frac{16}{15}+\frac{1}{6}=\frac{5}{3}=1\frac{2}{3}$$

21. 

[−5, 5] by [−1, 14]

The curves intersect at $x=0$ and $x=\pm4$. Because of the absolute value sign, break the integral up at $x=\pm2$ also (where $\left|x^2-4\right|$ turns the corner). Use the graph's symmetry:

$$2\int_0^2\left[\left(\frac{x^2}{2}+4\right)-(4-x^2)\right]dx+2\int_2^4\left[\left(\frac{x^2}{2}+4\right)-(x^2-4)\right]dx$$
$$=2\int_0^2\frac{3x^2}{2}\,dx+2\int_2^4\left(-\frac{x^2}{2}+8\right)dx$$
$$=2\left[\frac{x^3}{2}\right]_0^2+2\left[-\frac{x^3}{6}+8x\right]_2^4$$
$$=2[4]+2\left[\left(-\frac{32}{3}+32\right)-\left(-\frac{4}{3}+16\right)\right]=\frac{64}{3}=21\frac{1}{3}$$

22. Solve $y^2 = y+2$: $y^2 - y - 2 = (y-2)(y+1) = 0$, so the curves intersect at $y=-1$ and $y=2$.

$$\int_{-1}^{2}(y+2-y^2)\,dy = \left[\frac{1}{2}y^2+2y-\frac{1}{3}y^3\right]_{-1}^{2}$$
$$= \left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)$$
$$= \frac{9}{2} = 4\frac{1}{2}$$

23. Solve for x: $x = \frac{y^2}{4}-1$ and $x = \frac{y}{4}+4$.

Now solve $\frac{y^2}{4}-1 = \frac{y}{4}+4$: $\frac{y^2}{4}-\frac{y}{4}-5=0$,

$y^2 - y - 20 = (y-5)(y+4) = 0$.

The curves intersect at $y=-4$ and $y=5$.

$$\int_{-4}^{5}\left[\left(\frac{y}{4}+4\right)-\left(\frac{y^2}{4}-1\right)\right]dy$$
$$= \int_{-4}^{5}\left(-\frac{y^2}{4}+\frac{y}{4}+5\right)dy$$
$$= \left[-\frac{y^3}{12}+\frac{y^2}{8}+5y\right]_{-4}^{5}$$
$$= \left(-\frac{125}{12}+\frac{25}{8}+25\right)-\left(\frac{16}{3}+2-20\right) = \frac{243}{8} = 30\frac{3}{8}$$

24. Solve for x: $x = y^2$ and $x = 3-2y^2$. Now solve $y^2 = 3-2y^2$: $y^2 = 1$, so the curves intersect at $y=\pm 1$.

Use the region's symmetry:

$$2\int_0^1(3-2y^2-y^2)\,dy = 2\int_0^1(3-3y^2)\,dy$$
$$= 6\int_0^1(1-y^2)\,dy$$
$$= 6\left[y-\frac{1}{3}y^3\right]_0^1$$
$$= 6\left[\left(1-\frac{1}{3}\right)-0\right] = 4$$

25. Solve for x: $x=-y^2$ and $x = 2-3y^2$. Now solve $-y^2 = 2-3y^2$: $y^2=1$, so the curves intersect at $y=\pm 1$.

Use the region's symmetry:

$$2\int_0^1(2-3y^2+y^2)\,dy = 2\int_0^1(2-2y^2)\,dy$$
$$= 4\int_0^1(1-y^2)\,dy = 4\left[y-\frac{1}{3}y^3\right]_0^1 = 4\left[\left(1-\frac{1}{3}\right)-0\right] = \frac{8}{3}$$

26. Solve for y: $y = 4-4x^2$ and $y = x^4-1$.

Now solve $4-4x^2 = x^4-1$:

$x^4+4x^2-5 = (x^2-1)(x^2+5) = 0$.

The curves intersect at $x = \pm 1$.

Use the region's symmetry:

$$2\int_0^1\left[(4-4x^2)-(x^4-1)\right]dx$$
$$= 2\int_0^1(-x^4-4x^2+5)\,dx$$
$$= 2\left[-\frac{1}{5}x^5-\frac{4}{3}x^3+5x\right]_0^1$$
$$= 2\left[\left(-\frac{1}{5}-\frac{4}{3}+5\right)-0\right]$$
$$= \frac{104}{15} = 6\frac{14}{15}$$

27. Solve for x: $x = 3-y^2$ and $x = -\frac{y^2}{4}$.

Now solve $3-y^2 = -\frac{y^2}{4}$: $y^2 = 4$,

so the curves intersect at $y=\pm 2$.

Use the region's symmetry:

$$2\int_0^2\left(3-y^2+\frac{y^2}{4}\right)dy = 2\int_0^2\left(3-\frac{3y^2}{4}\right)dy$$
$$= 2\left[3y-\frac{y^3}{4}\right]_0^2$$
$$= 2(6-2)-0 = 8$$

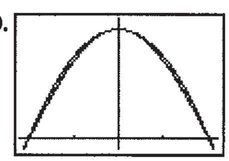
28. The curves intersect at 0 and π, so the area is:

$$\int_0^{\pi}(2\sin x - \sin 2x)\,dx = \left[-2\cos x+\frac{1}{2}\cos 2x\right]_0^{\pi}$$
$$= \left[\left(2+\frac{1}{2}\right)-\left(-2+\frac{1}{2}\right)\right] = 4$$

29. Use the region's symmetry:

$$2\int_0^{\pi/3}(8\cos x-\sec^2 x)\,dx = 2\left[8\sin x-\tan x\right]_0^{\pi/3}$$
$$= 2\left[(4\sqrt{3}-\sqrt{3})-0\right] = 6\sqrt{3}$$

30.

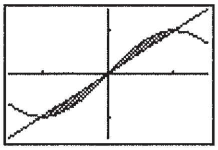


[-1.1, 1.1] by [-0.1, 1.1]

The curves intersect at $x=0$ and $x=\pm 1$, but they do not cross at $x=0$.

$$2\int_0^1\left[1-x^2-\cos\left(\frac{\pi x}{2}\right)\right]dx$$
$$= 2\left[x-\frac{1}{3}x^3-\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^1$$
$$= 2\left[\left(1-\frac{1}{3}-\frac{2}{\pi}\right)-0\right] = \frac{4}{3}-\frac{4}{\pi} \approx 0.0601$$

31.



[–1.5, 1.5] by [–1.5, 1.5]

The curves intersect at $x=0$ and $x=\pm 1$. Use the area's symmetry:

$$2\int_0^1\left[\sin\left(\frac{\pi x}{2}\right)-x\right]dx=2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)-\frac{1}{2}x^2\right]_0^1$$
$$=2\left[-\frac{1}{2}-\left(-\frac{2}{\pi}\right)\right]$$
$$=\frac{4-\pi}{\pi}\approx 0.273$$

32. Use the region's symmetry, and simplify before intergrating:

$$2\int_0^{\pi/4}(\sec^2 x-\tan^2 x)\,dx$$
$$=2\int_0^{\pi/4}\left[\sec^2 x-(\sec^2 x-1)\right]dx$$
$$=2\int_0^{\pi/4}dx=2\left[x\right]_0^{\pi/4}=\frac{\pi}{2}$$

33. Use the region's symmetry:

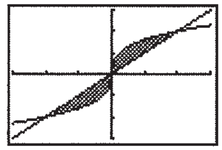
$$2\int_0^{\pi/4}(\tan^2 y+\tan^2 y)\,dy=4\int_0^{\pi/4}\tan^2 y\,dy$$
$$=4\left[\tan y-y\right]_0^{\pi/4}$$
$$=4\left[\left(1-\frac{\pi}{4}\right)-0\right]$$
$$=4-\pi\approx 0.858$$

34. $$\int_0^{\pi/2}3\sin y\sqrt{\cos y}\,dy=3\left[-\frac{2}{3}(\cos y)^{3/2}\right]_0^{\pi/2}$$
$$=3\left[0-\left(-\frac{2}{3}\right)\right]=2$$

35. $$\int_{-3}^1\sqrt{x+3}\,dx-\int_0^1(2x)\,dx$$
$$=\frac{2}{3}(x+3)^{3/2}\Big|_3^1-x^2\Big|_0^1$$
$$=\frac{2}{3}(1+3)^{3/2}-\frac{2}{3}(-3+3)^{3/2}-(1^2-0^2)$$
$$=4.333$$

36. $$\int_{-2}^1(4-x^2)\,dx-\int_0^1(3x)\,dx$$
$$=\left(4x-\frac{x^3}{3}\right)\Big|_{-2}^1-\frac{3x^2}{2}\Big|_0^1$$
$$=4-\frac{1^3}{3}-\left(4(-2)-\frac{-2^3}{3}\right)-\left(\frac{3(1)^2}{2}-0\right)$$
$$=15/2$$

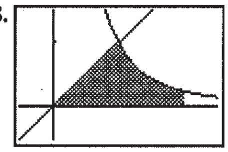
37. Solve for x: $x=y^3$ and $x=y$.



[–1.5, 1.5] by [–1.5, 1.5]

The curves intersect at $x=0$ and $x=\pm 1$. Use the area's symmetry: $2\int_0^1(y-y^3)\,dy=2\left[\frac{1}{2}y^2-\frac{1}{4}y^4\right]_0^1=\frac{1}{2}$

38.



[–0.5, 2.5] by [–0.5, 1.5]

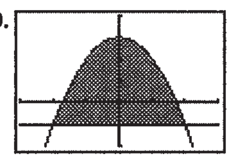
$y=x$ and $y=\dfrac{1}{x^2}$ intersect at $x=1$. Integrate in two parts:

$$\int_0^1 x\,dx+\int_1^2\frac{1}{x^2}\,dx=\left[\frac{1}{2}x^2\right]_0^1+\left[-\frac{1}{x}\right]_1^2$$
$$=\frac{1}{2}+\left[-\frac{1}{2}-(-1)\right]=1$$

39. The curves intersect when $\sin x=\cos x$, i.e., at $x=\dfrac{\pi}{4}$.

$$\int_0^{\pi/4}(\cos x-\sin x)\,dx=\left[\sin x+\cos\right]_0^{\pi/4}$$
$$=\sqrt{2}-1\approx 0.414$$

40.



[–3, 3] by [–2, 4]

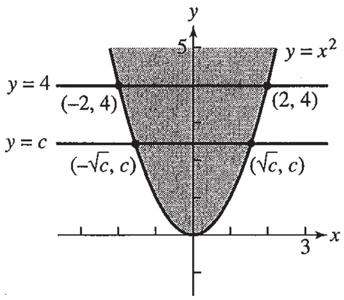
(a) The curves intersect at $x=\pm 2$.
Use the region's symmetry:

$$2\int_0^2(3-x^2+1)\,dx=2\int_0^2(4-x^2)\,dx$$
$$=2\left[4x-\frac{1}{3}x^3\right]_0^2$$
$$=2\left[\left(8-\frac{8}{3}\right)-0\right]=\frac{32}{3}$$

(b) Solve $y=3-x^2$ for x: $x=\pm\sqrt{3-y}$. The y-intercepts are –1 and 3.

$$\int_{-1}^3 2\sqrt{3-y}\,dy=2\left[-\frac{2}{3}(3-y)^{3/2}\right]_{-1}^3$$
$$=2\left[0-\left(-\frac{16}{3}\right)\right]=\frac{32}{3}$$

41. **(a)**



If $y = x^2 = c$, then $x = \pm\sqrt{c}$. So the points are $(-\sqrt{c},\, c)$ and $(\sqrt{c},\, c)$.

(b) The two areas in Quadrant I, where $x = \sqrt{y}$, are equal:

$$\int_0^c \sqrt{y}\,dy = \int_c^4 \sqrt{y}\,dy$$

$$\left[\frac{2}{3}y^{3/2}\right]_0^c = \left[\frac{2}{3}y^{3/2}\right]_c^4$$

$$\frac{2}{3}c^{3/2} = \frac{2}{3}4^{3/2} - \frac{2}{3}c^{3/2}$$

$$2c^{3/2} = 8$$

$$c^{3/2} = 4$$

$$c = 4^{2/3} = 2^{4/3}$$

(c) Divide the upper right section into a $(4 - c)$-by-$\sqrt{c}$ rectangle and a leftover portion:

$$\int_0^{\sqrt{c}} (c - x^2)\,dx = (4 - c)\sqrt{c} + \int_{\sqrt{c}}^2 (4 - x^2)\,dx$$

$$\left[cx - \frac{1}{3}x^3\right]_0^{\sqrt{c}} = 4\sqrt{c} - c^{3/2} + \left[4x - \frac{1}{3}x^3\right]_{\sqrt{c}}^2$$

$$c^{3/2} - \frac{1}{3}c^{3/2} = 4\sqrt{c} - c^{3/2} + \left[\left(8 - \frac{8}{3}\right) - \left(4\sqrt{c} - \frac{1}{3}c^{3/2}\right)\right]$$

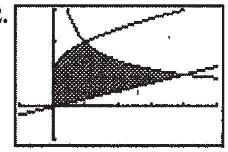
$$\frac{2}{3}c^{3/2} = 4\sqrt{c} - c^{3/2} + \frac{16}{3} - 4\sqrt{c} + \frac{1}{3}c^{3/2}$$

$$\frac{4}{3}c^{3/2} = \frac{16}{3}$$

$$c^{3/2} = 4$$

$$c = 4^{2/3} = 2^{4/3}$$

42.



[−1, 5] by [−1, 3]

The key intersection points are at $x = 0$, $x = 1$ and $x = 4$. Integrate in two parts:

$$\int_0^1 \left(1 + \sqrt{x} - \frac{x}{4}\right)dx + \int_1^4 \left(\frac{2}{\sqrt{x}} - \frac{x}{4}\right)dx$$

$$= \left[x + \frac{2}{3}x^{3/2} - \frac{x^2}{8}\right]_0^1 + \left[4\sqrt{x} - \frac{x^2}{8}\right]_1^4$$

$$= \left(1 + \frac{2}{3} - \frac{1}{8}\right) + \left[(8 - 2) - \left(4 - \frac{1}{8}\right)\right] = \frac{11}{3}$$

43. First find the two areas.

For the triangle, $\frac{1}{2}(2a)(a^2) = a^3$

For the parabola, $2\int_0^a (a^2 - x^2)\,dx = 2\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}a^3$

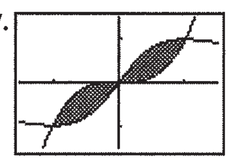
The ratio, then, is $\dfrac{a^3}{\frac{4}{3}a^3} = \dfrac{3}{4}$, which remains constant as a approaches zero.

44. $\int_a^b [2f(x) - f(x)]\,dx = \int_a^b f(x)\,dx$, which we already know equals 4.

45. Neither; both integrals come out as zero because the −1-to-0 and 0-to-1 portions of the integrals cancel each other.

46. Sometimes true, namely when $dA = [f(x) - g(x)]\,dx$ is always nonnegative. This happens when $f(x) \geq g(x)$ over the entire interval.

47.



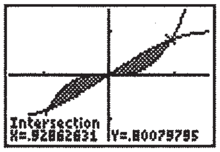
[−1.5, 1.5] by [−1.5, 1.5]

The curves intersect at $x = 0$ and $x = \pm 1$. Use the area's symmetry:

$$2\int_0^1 \left(\frac{2x}{x^2 + 1} - x^3\right)dx = 2\left[\ln(x^2 + 1) - \frac{1}{4}x^4\right]_0^1$$

$$= 2\ln 2 - \frac{1}{2}$$

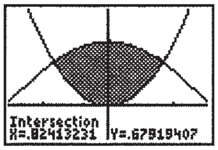
$$= \ln 4 - \frac{1}{2} \approx 0.886$$

48. 

[−1.5, 1.5] by [−1.5, 1.5]

The curves intersect at $x = 0$ $x \approx \pm 0.9286$. Use NINT to find $2\int_0^{0.9286} (\sin x - x^3)\,dx \approx 0.4303$.

49. First graph $y = \cos x$ and $y = x^2$.



[−1.5, 1.5] by [−0.5, 1.5]

The curves intersect at $x \approx \pm 0.8241$. Use NINT to find $2\int_0^{0.8241} (\cos x - x^2)\,dx \approx 1.0948$. Multiplying both functions by k will not change the x-value of any intersection point, so the area condition to be met is

$$2 = 2\int_0^{0.8241} (k\cos x - kx^2)\,dx$$
$$\Rightarrow 2 = k \cdot 2\int_0^{0.8241} (\cos x - x^2)\,dx$$
$$\Rightarrow 2 \approx k(1.0948)$$
$$\Rightarrow k \approx 1.8269.$$

50. True. 36 is the value of the appropriate integral.

51. False. It is $\int_0^{0.739} (\cos x - x)\,dx$.

52. A.

53. E. $\int_0^3 (x^2 - (-x))\,dx = \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^3 = \frac{27}{2}$

54. B. The curves intersect at $x \approx 1.139$. Use NINT to find

$$\int_0^{1.139} \left(e^{-x^2} - (-\sin 3x)\right)dx \approx 1.445$$

55. A. $\int_1^2 \left(e^x - \frac{1}{x}\right)dx$

$$= \left(e^x - \ln x\right)\Big|_1^2$$
$$= e^2 - \ln 2 - e$$

56. (a) Solve for y:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$
$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

(b) $\int_{-a}^{a} \left[b\sqrt{1-\frac{x^2}{a^2}} - \left(-b\sqrt{1-\frac{x^2}{a^2}}\right)\right]dx$ or

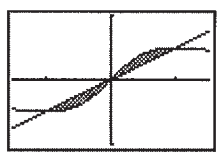
$2\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx$ or $4\int_0^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx$

(c) Answers may vary.

(d, e) $$2\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx = 2b\left[\frac{x}{2}\sqrt{1-\frac{x^2}{a^2}} + \frac{a}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}$$
$$= 2b\left[\frac{a}{2}\sin^{-1}(1) - \frac{a}{2}\sin^{-1}(-1)\right]$$
$$= \pi ab$$

57. By hypothesis, $f(x) - g(x)$ is the same for each region, where $f(x)$ and $g(x)$ represent the upper and lower edges. But then Area $= \int_a^b \left[f(x) - g(x)\right]\,dx$ will be the same for each.

58. The curves are shown for $m = \frac{1}{2}$:



[−1.5, 1.5] by [−1, 1]

In general, the intersection points are where $\frac{x}{x^2+1} = mx$, which is where $x = 0$ or else $x = \pm\sqrt{\frac{1}{m} - 1}$. Then, because of symmetry, the area is

$$2\int_0^{\sqrt{(1/m)-1}} \left(\frac{x}{x^2+1} - mx\right)dx$$
$$= 2\left[\frac{1}{2}\ln(x^2+1) - \frac{1}{2}mx^2\right]_0^{\sqrt{(1/m)-1}}$$
$$= \ln\left(\frac{1}{m} - 1 + 1\right) -$$
$$m\left(\frac{1}{m} - 1\right) = m - \ln(m) - 1.$$