Section 10.3 Polar Functions (pp. 548–559)

Exploration 1 Graphing Polar Curves Parametrically

See text page 551.

Quick Review 10.3

1. $x = 4\ \cos 30 = 2\sqrt{3}$

$y = 4\ \sin 30 = 2$

$\langle 2\sqrt{3},\ 2\rangle$

2. $A = \pi r^2 \dfrac{30}{360} = \pi(6)^2 \dfrac{30}{360} = 3\pi$

3. $A = \pi r^2\ \dfrac{\pi}{8}\left(\dfrac{1}{2\pi}\right) = \dfrac{1}{16}\pi\ (8)^2 = 4\pi$

4. $x^2 + y^2 = 25$

5. Graph $y = \left(\dfrac{4-x^2}{3}\right)^{1/2}$ and $y = -\left(\dfrac{4-x^2}{3}\right)^{1/2}$

6. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{d/dt(5\sin t)}{d/dt(3\cos t)}$

$= \dfrac{5\cos t}{-3\sin t} = -\dfrac{5}{3}\cot t$

7. $-\dfrac{5}{3}\cot(2) = 0.763$

8. $-\dfrac{5}{3}\cot t = 0,\quad$ so $\ t = \dfrac{\pi}{2}\ $ or $\ \dfrac{3\pi}{2}$

$x = 3\cos\dfrac{\pi}{2} = 0 \quad x = 3\cos\dfrac{3\pi}{2} = 0$

$y = 5\sin\dfrac{\pi}{2} = 5 \quad y = 5\sin\dfrac{3\pi}{2} = -5$

(0, 5) and (0, –5)

9. $-3\sin t = 0,\quad$ so $\ t = 0\ $ or $\ t = \pi$

$x = 3\cos 0 = 3 \quad x = 3\cos\pi = -3$

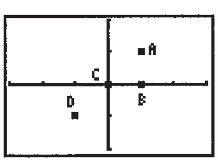
$y = 5\sin 0 = 0 \quad y = 5\sin\pi = 0$

(3, 0) and (–3, 0)

10. $\int_0^{\pi} \sqrt{(3\cos t)^2 + (5\sin t)^2}\,dt \approx 12.763$

Section 10.3 Exercises

1.



[–3, 3] by [–2, 2]

(a) $x = \sqrt{2}\ \cos\left(\dfrac{\pi}{4}\right) = 1$

$y = \sqrt{2}\sin\left(\dfrac{\pi}{4}\right) = 1$

(1, 1)

(b) $x = 1\cos(0) = 1$

$y = 1\sin(0) = 0$

$(1, 0)$

(c) $x = 0\cos\left(\dfrac{\pi}{2}\right) = 0$

$y = 0\sin\left(\dfrac{\pi}{2}\right) = 0$

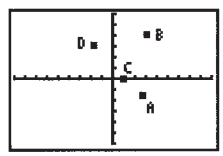
(0, 0)

(d) $x = -\sqrt{2}\cos\left(\dfrac{\pi}{4}\right) = -1$

$y = -\sqrt{2}\ \sin\left(\dfrac{\pi}{4}\right) = -1$

$(-1, -1)$

2.



[–9, 9] by [–6, 6]

(a) $x = -3\cos\left(\dfrac{5\pi}{6}\right) = \dfrac{3\sqrt{3}}{2}$

$y = -3\sin\left(\dfrac{5\pi}{6}\right) = \dfrac{-3}{2}$

$\left(\dfrac{3\sqrt{3}}{2}, \dfrac{-3}{2}\right)$

(b) $x = 5\cos\left(\tan^{-1}\left(\dfrac{4}{3}\right)\right) = 3$

$y = 5\sin\left(\tan^{-1}\left(\dfrac{4}{3}\right)\right) = 4$

$(3, 4)$

(c) $x = -1\cos 7\pi = 1$

$y = -1\sin 7\pi = 0$

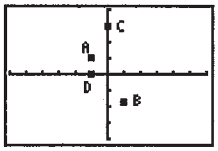
$(1, 0)$

(d) $x = 2\sqrt{3}\cos\left(\dfrac{2\pi}{3}\right) = -\sqrt{3}$

$y = 2\sqrt{3}\sin\left(\dfrac{2\pi}{3}\right) = 3$

$\left(-\sqrt{3}, 3\right)$

3.



[–6, 6] by [–4, 4]

(a) $r=\sqrt{(-1)^2+1^2}=\sqrt{2}$

$\theta=\tan^{-1}\left(\dfrac{1}{-1}\right)=-\dfrac{5\pi}{4},\dfrac{3\pi}{4}$

$\left(\sqrt{2},\dfrac{-5\pi}{4}\right)$ and $\left(\sqrt{2},\dfrac{3\pi}{4}\right)$

(b) $r=\sqrt{1+\left(-\sqrt{3}\right)^2}=\pm 2$

$\theta=\tan^{-1}\left(\dfrac{-\sqrt{3}}{1}\right)=-\dfrac{\pi}{3},\dfrac{2\pi}{3}$

$\left(-2,\dfrac{2\pi}{3}\right)$ and $\left(2,-\dfrac{\pi}{3}\right)$

(c) $r=\sqrt{0^2+3^2}=\pm 3$

$\theta=\tan^{-1}\left(\dfrac{3}{0}\right)=\dfrac{\pi}{2},\dfrac{5\pi}{2}$

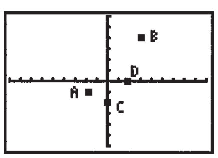
$\left(3,\dfrac{\pi}{2}\right)$ and $\left(3,\dfrac{5\pi}{2}\right)$

(d) $r=\sqrt{\left(-1\right)^2+0^2}=\pm 1$

$\theta=\tan^{-1}\left(\dfrac{0}{1}\right)=0,\pi$

$(-1,0)$ and $(1,\pi)$

4.



[–9, 9] by [–6, 6]

(a) $r=\sqrt{\left(-\sqrt{3}\right)^2+(-1)^2}=\pm 2$

$\theta=\tan^{-1}\left(\dfrac{-1}{-\sqrt{3}}\right)=\dfrac{\pi}{6},\dfrac{7\pi}{6}$

$\left(-2,\dfrac{\pi}{6}\right)$ and $\left(2,\dfrac{7\pi}{6}\right)$

(b) $r=\sqrt{3^2+4^2}=\pm 5$

$\theta=\tan^{-1}\left(\dfrac{4}{3}\right)$

$\left(-5,\pi+\tan^{-1}\left(\dfrac{4}{3}\right)\right)$ and $\left(5,\tan^{-1}\left(\dfrac{4}{3}\right)\right)$

(c) $r=\sqrt{0^2+(-2)^2}=\pm 2$

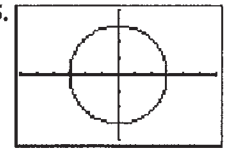
$\theta=\tan^{-1}\left(\dfrac{-2}{0}\right)=-\dfrac{\pi}{2},\dfrac{3\pi}{2},\left(2,-\dfrac{\pi}{2}\right)$ and $\left(2,\dfrac{3\pi}{2}\right)$

(d) $r=\sqrt{2^2+0^2}=\pm 2$

$\theta=\tan^{-1}\left(\dfrac{0}{2}\right)=0,2\pi$

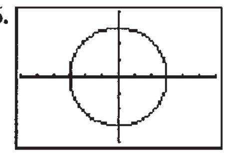
$(2,0)$ and $(2,2\pi)$

5.



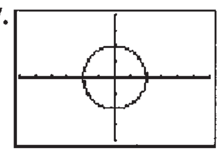
[–6, 6] by [–4, 4]
$0\le\theta\le 2\pi$

6.



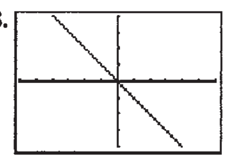
[–6, 6] by [–4, 4]
$0\le\theta\le 2\pi$

7.



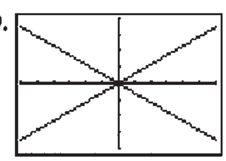
[–6, 6] by [–4, 4]
$0\le\theta\le 2\pi$

8.



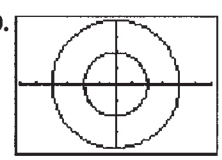
[–6, 6] by [–4, 4]

9.



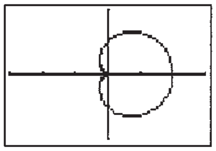
[–6, 6] by [–4, 4]

10.



[–6, 6] by [–4, 4]
$0\le\theta\le 2\pi$

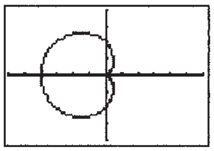
11. Cardioid



$[-3, 3]$ by $[-2, 2]$

$0 \le \theta \le 2\pi$

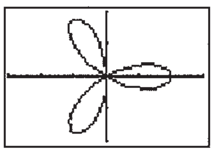
12. Cardioid



$[-6, 6]$ by $[-4, 4]$

$0 \le \theta \le 2\pi$

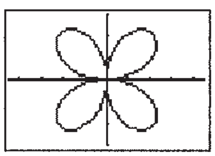
13. Rose



$[-3, 3]$ by $[-2, 2]$

$0 \le \theta \le \pi$

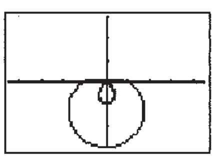
14. Rose



$[-4.5, 4.5]$ by $[-3, 3]$

$0 \le \theta \le 2\pi$

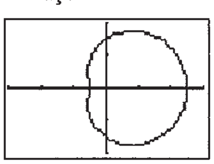
15. Limaçon



$[-4.5, 4.5]$ by $[-3, 3]$

$0 \le \theta \le 2\pi$

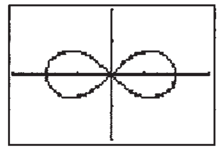
16. Limaçon



$[-3, 3]$ by $[-2, 2]$

$0 \le \theta \le 2\pi$

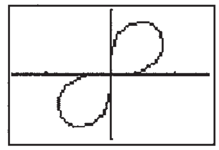
17. Lemniscate



$[-3, 3]$ by $[-2, 2]$

$-\pi/4 \le \theta \le \pi/4$

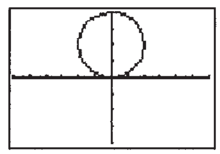
18. Lemniscate



$[-1.5, 1.5]$ by $[-1, 1]$

$0 \le \theta \le \pi/2$

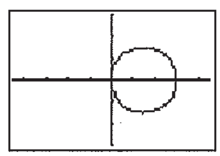
19. Circle



$[-6, 6]$ by $[-4, 4]$

$0 \le \theta \le \pi$

20. Circle



$[-4.5, 4.5]$ by $[-3, 3]$

$0 \le \theta \le \pi$

21. $r = 4\csc\theta$

$$1 = \frac{4}{r\sin\theta}$$

$y = 4$, a horizontal line

22. $r = -3\sec\theta$

$$1 = \frac{-3}{r\cos\theta}$$

$x = -3$, a vertical line

23. $x + y = 1$, a line

(slope $= -1$, y-intercept $= 1$)

24. $r^2 = x^2 + y^2 = 1$,

a circle (center $= (0, 0)$, radius $= 1$)

25. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

$1 = \dfrac{5}{r\sin\theta - 2r\cos\theta}$

$y - 2x = 5$, a line (slope = 2, y-intercept = 5)

26. $r^2 \sin 2\theta = 2$

$r^2(2\sin\theta\cos\theta) = 2$

$2xy = 2$

$xy = 1$, a hyperbola

27. $r^2\cos^2\theta = r^2\sin\theta$

$x^2 = y^2$, the union of two lines: $y = \pm x$

28. $r^2 = -4r\cos\theta$

$x^2 + y^2 = -4x$

$x^2 + 4x + 4 - 4 + y^2 = 0$

$(x+2)^2 + y^2 = 4$, a circle (center = (–2, 0), radius = 2)

29. $r^2 = 8r\sin\theta$

$x^2 + y^2 - 8r\sin\theta + 16 - 16 = 0$

$x^2 + (y-4)^2 = 16$, a circle (center = (0, 4), radius = 4)

30. $r = 2\cos\theta + 2\sin\theta$

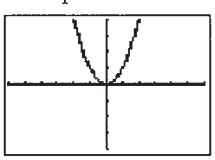
$r^2 = 2r\cos\theta + 2r\sin\theta$

$x^2 + y^2 = 2x + 2y$

$(x-1)^2 + (y-1)^2 = 2$,

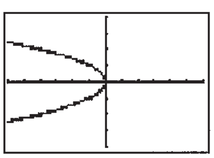
a circle $\left(\text{center} = (1, 1), \text{radius} = \sqrt{2}\right)$

31. It is a parabola.



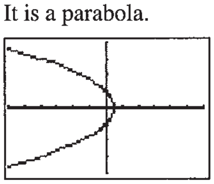
[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

32. It is a parabola.



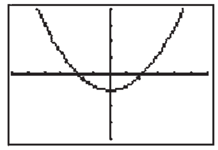
[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

33. It is a parabola.



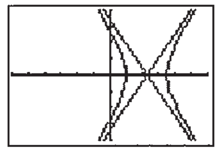
[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

34. It is a parabola.



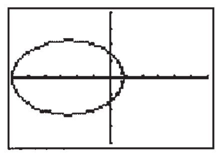
[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

35. It is a hyperbola.



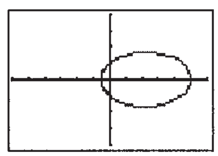
[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

36. It is an ellipse.



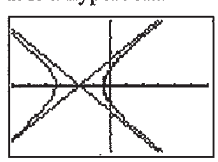
[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

37. It is an ellipse.



[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

38. It is a hyperbola.



[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

39. $r = -1 + \sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}((-1+\sin\theta)\sin\theta)}{\frac{d}{d\theta}((-1+\sin\theta)\cos\theta)}$$

$$\frac{dy}{dx} = \frac{(2\sin\theta - 1)\cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}$$

At $\theta = 0$: -1

At $\theta = \pi$: 1

40. $r = \cos 2\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(\cos 2\theta \sin\theta)}{\frac{d}{d\theta}(\cos 2\theta \cos\theta)}$$

$$\frac{dy}{dx} = \frac{\cos\theta \cos 2\theta - 2\sin\theta \sin 2\theta}{-\sin\theta \cos 2\theta - 2\cos\theta \sin 2\theta}$$

At $\theta = 0$: undefined

At $\theta = -\frac{\pi}{2}$: 0

At $\theta = \frac{\pi}{2}$: 0

At $\theta = \pi$: undefined

41. $r = 2 - 3\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(2 - 3\sin\theta)\sin\theta}{\frac{d}{d\theta}(2 - 3\sin\theta)\cos\theta}$$

$$\frac{dy}{dx} = \frac{(2 - 6\sin\theta)\cos\theta}{\sin\theta(3\sin\theta - 2) - 3\cos^2\theta}$$

At $(2, 0)$: $-\frac{2}{3}$

At $\left(-1, \frac{\pi}{2}\right)$: 0

At $(2, \pi)$: $\frac{2}{3}$

At $\left(5, \frac{3\pi}{2}\right)$: 0

42. $r = 3(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}\big(3(1 - \cos\theta)\sin\theta\big)}{\frac{d}{d\theta}\big(3(1 - \cos\theta)\cos\theta\big)}$$

$$\frac{dy}{dx} = \frac{-6\cos^2\theta + 3\cos\theta + 3}{3\sin\theta(2\cos\theta - 1)}$$

At $\left(1.5, \frac{\pi}{3}\right)$: undefined

At $\left(4.5, \frac{2\pi}{3}\right)$: 0

At $(6, \pi)$: undefined

At $\left(3, \frac{3\pi}{2}\right)$: 1

43. $\int_0^{2\pi} \frac{1}{2}(4 + 2\cos\theta)^2 \, d\theta$

$$= \int_0^{2\pi} \frac{1}{2}(16 + 16\cos\theta + 4\cos^2\theta)d\theta$$

$$= \int_0^{2\pi} (8 + 8\cos\theta + 1 + \cos 2\theta)d\theta$$

$$= \left[9\theta + 8\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 18\pi$$

44. $\int_0^{2\pi} \frac{1}{2}(2 + 2\sin\theta)^2 \, d\theta$

$$= \int_0^{2\pi} \frac{1}{2}(4 + 8\sin\theta + 4\sin^2\theta)d\theta$$

$$= \int_0^{2\pi} (2 + 4\sin\theta + 1 - \cos 2\theta)d\theta$$

$$= \left[3\theta - 4\cos\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 6\pi - 4 + 4 = 6\pi$$

45. $\int_{-\pi/4}^{\pi/4} \frac{1}{2}(\cos 2\theta)^2 \, d\theta$

$$= \int_0^{\pi/4} \cos^2(2\theta)d\theta$$

$$= \int_0^{\pi/4} \left(\frac{1}{2} + \frac{\cos 4\theta}{2}\right)d\theta$$

$$= \left[\frac{\theta}{2} + \frac{\sin 4\theta}{8}\right]_0^{\pi/4}$$

$$= \frac{\pi}{8}$$

46. $\int_0^{2\pi} \frac{1}{2}(2\sin 4\theta)^2 \, d\theta$

$$= \int_0^{2\pi} 2\sin^2(4\theta)d\theta$$

$$= \int_0^{2\pi} (1 - \cos 8\theta)d\theta$$

$$= \left[\theta - \frac{\sin 8\theta}{8}\right]_0^{2\pi}$$

$$= 2\pi$$

47. $\int_{-\pi/4}^{\pi/4} \frac{1}{2}(4\cos 2\theta)d\theta$

$$= \sin 2\theta \Big|_{-\pi/4}^{\pi/4}$$

$$= 2$$

48. $6\int_0^{\pi/3} \frac{1}{2}(2\sin 3\theta)d\theta$

$$= -2\cos 3\theta \Big|_0^{\pi/3}$$

$$= 2 - (-2) = 4$$

49. $\int_0^{2\pi} \frac{1}{2}(3-2\cos\theta)^2\,d\theta$

$= \int_0^{2\pi} \frac{1}{2}(9-12\cos\theta+4\cos^2\theta)\,d\theta$

$= \int_0^{2\pi} \left(\frac{9}{2}-6\cos\theta+1+\cos 2\theta\right)d\theta$

$= \left[\frac{11\theta}{2}-6\sin\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi}$

$= 11\pi - 0 = 11\pi$

50. $\int_{\pi/6}^{5\pi/6} \frac{1}{2}(2\sin\theta-1)^2\,d\theta$

$= \int_{\pi/6}^{5\pi/6} \left(2\sin^2\theta-2\sin\theta+\frac{1}{2}\right)d\theta$

$= \int_{\pi/6}^{5\pi/6} \left(1-\cos 2\theta-2\sin\theta+\frac{1}{2}\right)d\theta$

$= \left[\frac{3\theta}{2}-\frac{\sin 2\theta}{2}+2\cos\theta\right]_{\pi/6}^{5\pi/6}$

$= \pi - \frac{3\sqrt{3}}{2} = 0.544$

51. $2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2 d\theta$

$= \int_0^{\pi/4} 4\sin^2\theta d\theta$

$= \int_0^{\pi/4} 2(1-\cos 2\theta)\,d\theta$

$= [2\theta - \sin 2\theta]_0^{\pi/4}$

$= \frac{\pi}{2} - 1$

52. $2\left(\int_0^{\pi/6} \frac{1}{2}(2\sin\theta)^2 d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(1)^2 d\theta\right)$

$= \int_0^{\pi/6} 4\sin^2\theta d\theta + \int_{\pi/6}^{\pi/2} 1 d\theta$

$= \int_0^{\pi/6} (2-2\cos 2\theta)\,d\theta + \frac{\pi}{3}$

$= [2\theta - \sin 2\theta]_0^{\pi/6} + \frac{\pi}{3}$

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

53. $2\left(\int_0^{\pi/2} \frac{1}{2}[2(1-\cos\theta)]^2 d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}(2)^2 d\theta\right)$

$= \int_0^{\pi/2} 4(1-2\cos\theta+\cos^2\theta)\,d\theta + \int_{\pi/2}^{\pi} 4d\theta$

$= \int_0^{\pi/2} (4-8\cos\theta+2(1+\cos 2\theta))\,d\theta + 2\pi$

$= [6\theta - 8\sin\theta + \sin 2\theta]_0^{\pi/2} + 2\pi$

$= 5\pi - 8$

54. $4\int_0^{\pi/2} \frac{1}{2}(2(1-\cos\theta))^2 d\theta$

$= 4\int_0^{\pi/2} (2-4\cos\theta+2\cos^2\theta)\,d\theta$

$= \int_0^{\pi/2} (8-16\cos\theta+4(1+\cos 2\theta))\,d\theta$

$= [12\theta - 16\sin\theta + 2\sin 2\theta]_0^{\pi/2}$

$= 6\pi - 16$

55. $\int_0^{\pi} \frac{1}{2}(2(1-\sin\theta))^2\,d\theta + \int_{\pi}^{2\pi} \frac{1}{2}(2)^2\,d\theta$

$= \int_0^{\pi} (2-4\sin\theta+2\sin^2\theta)\,d\theta + \int_{\pi}^{2\pi} 2d\theta$

$= \int_0^{\pi} (2-4\sin\theta+1-\cos 2\theta)\,d\theta + 2\pi$

$= \left[3\theta + 4\cos\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} + 2\pi$

$= 5\pi - 8$

56. $4\int_{-\pi/6}^{\pi/6} \frac{1}{2}((4\cos 2\theta)^2 - 2^2)\,d\theta$

$= 4\int_0^{\pi/6} (16\cos^2 2\theta - 4)\,d\theta$

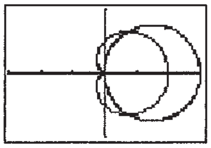
$= 4\int_0^{\pi/6} (8(1+\cos 4\theta)-4)\,d\theta$

$= \int_0^{\pi/6} (16+32\cos 4\theta)\,d\theta$

$= [16\theta + 8\sin 4\theta]_0^{\pi/6}$

$= \frac{8\pi}{3} + 4\sqrt{3}$

57.

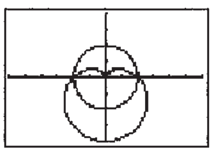


[–3, 3] by [–2, 2]

$0 \le \theta \le \pi$ for the circle
$0 \le \theta \le 2\pi$ for the cardioid

$$2\int_0^{\pi/3} \frac{1}{2}((3\cos\theta)^2 - (1+\cos\theta)^2)d\theta$$
$$= \int_0^{\pi/3} (8\cos^2\theta - 1 - 2\cos\theta)d\theta$$
$$= \int_0^{\pi/3} (4(1+\cos 2\theta) - 1 - 2\cos\theta)d\theta$$
$$= [3\theta + 2\sin 2\theta - 2\sin\theta]_0^{\pi/3}$$
$$= \pi + \sqrt{3} - \sqrt{3} - 0$$
$$= \pi$$

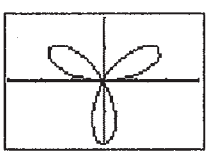
58.



[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

$$\int_0^{\pi} \frac{1}{2}(2^2 - (2(1-\sin\theta))^2)d\theta$$
$$= \int_0^{\pi} (2 - 2 + 4\sin\theta - 2\sin^2\theta)d\theta$$
$$= \int_0^{\pi} (4\sin\theta - 1 + \cos 2\theta)d\theta$$
$$= \left[-4\cos\theta - \theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$
$$= 4 - \pi - (-4)$$
$$= 8 - \pi$$

59.



[–3, 3] by [–2, 2]
$0 \le \theta \le \pi$

$$\int_0^{\pi} \frac{1}{2}(2\sin 3\theta)^2 d\theta$$
$$= \int_0^{\pi} 2\sin^2 3\theta d\theta$$
$$= \int_0^{\pi} (1 - \cos 6\theta)d\theta$$
$$= \left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi}$$
$$= \pi$$

$$\text{Slope} = \left.\frac{\frac{d}{d\theta}(2\sin 3\theta\sin\theta)}{\frac{d}{d\theta}(2\sin 3\theta\cos\theta)}\right|_{\theta=\pi/4}$$
$$= \left[\frac{6\sin\theta\cos 3\theta + 2\cos\theta\sin 3\theta}{6\cos\theta\cos 3\theta - 2\sin\theta\sin 3\theta}\right]_{\theta=\pi/4}$$
$$= \frac{1}{2}$$

60. (a) $\int_0^{3/4}\left(\sqrt{1+y^2} - \frac{5y}{3}\right]dy$

$$= \left[\frac{\ln\left(\left|\sqrt{y^2+1}+y\right|\right)}{2} + \frac{y\sqrt{y^2+1}}{2} - \frac{5y^2}{6}\right]_0^{3/4}$$
$$= 0.347$$

(b) $x = r\cos\theta$
$y = r\sin\theta$
$x^2 = 1 + y^2$
$x^2 - y^2 = 1$
$r^2\left(\cos^2\theta - \sin^2\theta\right) = 1$
$r^2 = \dfrac{1}{\cos^2\theta - \sin^2\theta}$

(c) Let $\alpha = \tan^{-1}(3/5)$.
Then the area is
$$\int_0^{\alpha} \frac{1}{2}\left(\frac{1}{\cos^2\theta - \sin^2\theta}\right)d\theta.$$

61. True. Polar coordinates determine a unique point.

62. False. Integrating from 0 to 2π traverses the curve twice, giving twice the area. The correct upper limit of integration is π.

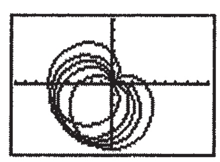
63. D.

64. E.

65. B.

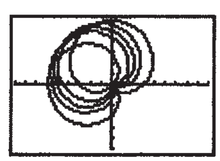
66. D.

67. (a)



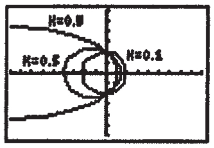
[–9, 9] by [–6, 6]

(b)



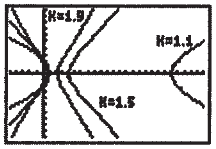
[–9, 9] by [–6, 6]

(c) The graph of r_2 is the graph of r_1 rotated counterclockwise about the origin by angle α.

68. (a) 

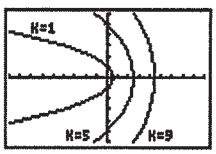
[−9, 9] by [−6, 6]
The graphs are ellipses.

(b) As $k \to 0^+$, the graph approaches the circle of radius 2 centered at the origin.

69. (a) 

[−5, 25] by [−10, 10]
The graphs are hyperbolas.

(b) As $k \to 1^+$, the right branch of the hyperbola goes to infinity and "disappears." The left branch approaches the parabola $y^2 = 4 - 4x$.

70. (a) 

[−9, 9] by [−6, 6]
The graphs are parabolas.

(b) As $k \to 0^+$, the limit of the graph is the negative x-axis.

71. $d = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]^{1/2}$

$$d = \left[(r_2 \cos\theta_2 - r_1 \cos\theta_1)^2 + (r_2 \sin\theta_2 - r_1 \sin\theta_1)^2\right]^{1/2}$$

$$d = \left[r_2^2 \cos^2\theta_2 + r_1^2 \cos^2\theta_1 + r_2^2 \sin^2\theta_2 + r_1^2 \sin^2\theta_1 + 2 r_1 r_2 \cos\theta_1 \cos\theta_2 + 2 r_1 r_2 \sin\theta_1 \sin\theta_2\right]^{1/2}$$

$$d = \left[r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_1 - \theta_2)\right]^{1/2}$$

72. (a) $\dfrac{1}{2\pi - 0}\displaystyle\int_0^{2\pi} a(1 - \cos\theta)\, d\theta = \dfrac{a}{2\pi}\left[\theta - \sin\theta\right]_0^{2\pi} = a$

(b) $\dfrac{1}{2\pi - 0}\displaystyle\int_0^{2\pi} a\, d\theta = \dfrac{a}{2\pi}\left[\theta\right]_0^{2\pi} = a$

(c) $\dfrac{1}{\pi/2 - (-\pi/2)}\displaystyle\int_{-\pi/2}^{\pi/2} a\cos\theta\, d\theta = \dfrac{a}{\pi}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = \dfrac{2a}{\pi}$

73.
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2$$
$$= (f'(\theta)\cos\theta)^2 + (f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta)^2 + (f(\theta)\cos\theta)^2$$
$$= (f(\theta))^2(\cos^2\theta + \sin^2\theta) + (f'(\theta))^2(\cos^2\theta + \sin^2\theta)$$
$$= (f(\theta))^2 + (f'(\theta))^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

74.
$$\int_0^{2\pi} \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{2\cos\theta + 2}\, d\theta$$
$$= 2\int_0^{\pi} \sqrt{2\cos\theta + 2}\, d\theta$$
$$= 2\int_0^{\pi} 2\cos\left(\frac{\theta}{2}\right) d\theta$$
$$= 8\sin\left(\frac{\theta}{2}\right)\Big]_0^{\pi}$$
$$= 8$$

Quick Quiz Sections 10.1–10.3

1. A.

2. C. $\dfrac{dx}{dt} = \dfrac{d}{dt}\left(t^3 - t^2 - 1\right) = 3t^2 - 2t$

$$3t^2 - 2t = 0$$
$$t(3t - 2) = 0$$
$$t = 0, 2/3$$

3. D.

4. (a) Area $= \dfrac{1}{2}\displaystyle\int_0^{\pi} (\theta + \sin 2\theta)^2\, d\theta = 4.382$

(b) $-2 = r\cos\theta = (\theta + \sin 2\theta)\cos\theta \Rightarrow \theta = 2.786$

(c) The graph is getting closer to the origin as θ increases from $\pi/3$ to $2\pi/3$.

(d) Maximize $r = \theta + \sin 2\theta$ for $0 \le \theta \le \dfrac{\pi}{2}$.

$$\frac{dr}{d\theta} = 1 + 2\cos 2\theta$$
$$1 + 2\cos 2\theta = 0$$
$$\cos 2\theta = -\frac{1}{2}$$
$$\theta = \frac{\pi}{3}$$

Since $\dfrac{dr}{d\theta} = 1 + 2\cos 2\theta > 0$ for $0 < \theta < \dfrac{\pi}{3}$ and $\dfrac{dr}{d\theta} = 1 + 2\cos 2\theta < 0$ for $\dfrac{\pi}{3} < \theta < \dfrac{\pi}{2}$, there is a maximum of r when $\theta = \dfrac{\pi}{3}$ by the First Derivative test. The curve is farthest from the origin when $\theta = \dfrac{\pi}{3}$.