2. If $0 < p < 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{k\to\infty}\int_1^k \frac{1}{x^p}\,dx$$
$$= \lim_{k\to\infty}\left(\left[\frac{x^{-p+1}}{-p+1}\right]_1^k\right)$$
$$= \lim_{k\to\infty}\left(\frac{1}{1-p}\cdot(k^{1-p}-1)\right)$$
$$= \infty \quad (\text{since } 1-p>0).$$

The series diverges by the Integral Test.
If $p \le 0$, the series diverges by the nth-Term Test.
This completes the proof for $p < 1$.

3. If $p = 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{k\to\infty}\int_1^k \frac{1}{x}\,dx$$
$$= \lim_{k\to\infty}\left(\ln x\big]_1^k\right)$$
$$= \lim_{k\to\infty} \ln k = \infty.$$

The series diverges by the Integral Test.

Exploration 2 The Maclaurin Series of a Strange Function

1. Since $f^{(n)}(0) = 0$ for all n, the Maclaurin Series for f has all zero coefficients! The series is simply $\sum_{n=0}^{\infty} 0 \cdot x^n = 0$.

2. The series converges (to 0) for all values of x.

3. Since $f(x) = 0$ only at $x = 0$, the only place that this series actually converges to $f(x)$ is at $x = 0$.

Quick Review 9.5

1. Converges, since it is of the form $\int_1^\infty \frac{1}{x^p}\,dx$ with $p > 1$.

2. Diverges, limit comparison test with integral of $\frac{1}{x}$.

3. Diverges, comparison test with integral of $\frac{1}{x}$.

4. Converges, comparison test with integral of $\frac{2}{x^2}$.

5. Diverges, limit comparison test with integral of $\frac{1}{\sqrt{x}}$.

6. Yes. $f(x) = \frac{3}{x} > 0$ for $x > 0$.

$f'(x) = -\frac{3}{x^2} < 0$ for $x \ne 0$.

Therefore f is positive and decreasing on $(0, \infty)$.

7. Yes. $f(x) = \frac{7x}{x^2-8} > 0$ when $x > 0$ and $x^2 > 8$, or when $x < 0$ and $x^2 < 8$.

$f'(x) = -\frac{7(x^2+8)}{(x^2-8)^2} < 0$ for $x^2 \ne 8$.

Therefore, f is positive and decreasing on $(2\sqrt{2}, \infty)$.

(Also on $(-2\sqrt{2}, 0)$, but that is not the kind of interval we are looking for.)

8. No. $f(x) = \frac{3+x^2}{3-x^2} > 0$ only when $3 - x^2 > 0$, so $f(x)$ is positive only for $-\sqrt{3} < x < \sqrt{3}$.

9. No. On any interval (N, ∞), $\sin x$ will oscillate between -1 and 1, so $f(x)$ will oscillate between positive and negative values, as well.

10. No. $x > 1$ implies $\frac{1}{x} < 1$, so $f(x) = \ln\left(\frac{1}{x}\right) < 0$ for all $x > 1$.

Section 9.5 Exercises

1. $$\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx = \lim_{k\to\infty}\int_1^k x^{-1/3}\,dx$$
$$= \lim_{k\to\infty}\left[\frac{3}{2}x^{2/3}\right]_1^k$$
$$= \lim_{k\to\infty}\left(\frac{3}{2}k^{2/3} - \frac{3}{2}(1)\right)$$
$$= \infty$$

Since $\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$ diverges, it follows from the Integral Test that $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ also diverges.

2. $$\int_1^\infty x^{-3/2}\,dx = \lim_{k\to\infty}\int_1^k x^{-3/2}\,dx$$
$$= \lim_{k\to\infty}\left[-\frac{1}{2}x^{-1/2}\right]_1^k$$
$$= \lim_{k\to\infty}\left(-\frac{1}{2\sqrt{k}} + \frac{1}{2\sqrt{1}}\right)$$
$$= 0 + \frac{1}{2}$$
$$= \frac{1}{2}$$

Since $\int_1^\infty x^{-3/2}\,dx$ converges, it follows from the Integral Test that $\sum_{n=1}^{\infty} n^{-3/2}$ also converges.

3. $S_1 = 1$

$$S_2 = 1+\frac{1}{2} = \frac{3}{2}$$
$$S_3 = \frac{3}{2}+\frac{1}{3} = \frac{11}{6}$$
$$S_4 = \frac{11}{6}+\frac{1}{4} = \frac{50}{24} = \frac{25}{12}$$
$$S_5 = \frac{25}{12}+\frac{1}{5} = \frac{137}{60}$$
$$S_6 = \frac{137}{60}+\frac{1}{6} = \frac{147}{60}$$

4. The quickest way is to use the calculator.

```
1→N:1→T
                    1
N+1→N:T+1/N→T
                  1.5
          1.833333333
          2.083333333
          2.283333333
```

Notice that T contains S_N.

Keep pressing ENTER until $T > 4$.

```
          3.891456753
          3.927171039
          3.961653798
          3.994987131
          4.027245195
N
                   31
```

31 is the first value of N for which $S_N > 4$.

Therefore, $k = 31$.

5. For n large, $\frac{3n-1}{n^2+1}$ behaves like $\frac{3}{n}$.

Let $a_n = \frac{3n-1}{n^2+1}$ and $b_n = \frac{1}{n}$.

Then $a_n > 0$ and $b_n > 0$ for $n > 0$, and

$$c = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(3n-1)/(n^2+1)}{1/n}$$
$$= \lim_{n\to\infty}\frac{n(3n-1)}{n^2+1} = \lim_{n\to\infty}\frac{3n^2-n}{n^2+1}$$
$$= \lim_{n\to\infty}\frac{3-\frac{1}{n}}{1+\frac{1}{n^2}} = 3$$

Since $0 < c < \infty$, and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (by the p-Test), $\sum_{n=1}^{\infty}\frac{3n-1}{n^2+1}$ diverges by the Limit Comparison Test.

6. For n large, $\frac{2^n}{3^n+1}$ behaves like $\frac{2^n}{3^n}$.

Let $a_n = \frac{2^n}{3^n+1}$ and $b_n = \frac{2^n}{3^n}$.

Then $a_n > 0$ and $b_n > 0$ for $n \ge 0$, and

$$c = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{[(2^n)/((3^n)+1)]}{\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty}\frac{2^n}{3^n+1}\cdot\frac{3^n}{2^n}$$
$$= \lim_{n\to\infty}\frac{3^n}{3^n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{3^n}} = 1$$

Since $0 < c < \infty$, and $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges (geometric series with $r = 2/3$), $\sum_{n=1}^{\infty}\frac{2^n}{3^n+1}$ converges by the Limit Comparison Test.

7. Diverges.

Method 1: Use the Integral Test with $\int_1^{\infty}\frac{5}{x+1}dx$.

Method 2: Use the Limit Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n}$ (which diverges by the p-test).

8. Diverges.

Method 1: Rewrite the series as $3\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ and use the p-test.

Method 2: Use the Integral Test with $\int_1^{\infty}\frac{3}{\sqrt{x}}dx$.

Method 3: Use the Direct Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (which diverges by the p-test).

Method 4: Use the Limit Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (which diverges by the p-test).

9. Diverges.

Method 1: Use the Integral Test with $\int_1^{\infty}\frac{\ln x}{x}dx$.

Method 2: Since $\ln n > 1$ for $n > 3$, use the Direct Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n}$ (which diverges by the p-test).

Method 3: Use the Limit Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n}$ (which diverges by the p-test).

10. Diverges.

Method 1: Use the Integral Test with $\int_1^\infty \frac{1}{2x-1}dx$.

Method 2: Use the Limit Comparison Test with $\sum_{n=1}^\infty \frac{1}{n}$ (which diverges by the p-test).

Method 3: Use the Direct Comparison Test:

$$2n-1<2n \Rightarrow \frac{1}{2n-1}>\frac{1}{2n}=\frac{1}{2}\bullet\frac{1}{n}.$$

Compare the given series with $\frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}$ (which diverges by the p-test).

11. Diverges. Geometric series with $r=\frac{1}{\ln 2}\approx 1.44$.

12. Converges. Geometric series with $r=\frac{1}{\ln 3}\approx 0.91$.

13. Diverges by the nth-Term Test:

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{\sin(1/n)}{1/n}=\lim_{x\to 0}\frac{\sin x}{x}=1\neq 0$$

14. Converges.

Method 1: Use the Integral Test:

$$\int_0^\infty \frac{e^x}{1+e^{2x}}dx=\lim_{k\to\infty}\tan^{-1}(e^x)\Big|_0^k$$
$$=\lim_{k\to\infty}\tan^{-1}(e^k)-\tan^{-1}(1)$$
$$=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

Method 2: Use the Direct Comparison Test:

$$\frac{e^n}{1+e^{2n}}<\frac{e^n}{e^{2n}}=\left(\frac{1}{e}\right)^n$$

Compare the given series with $\sum_{n=0}^\infty \left(\frac{1}{e}\right)^n$ (geometric series with $r=\frac{1}{e}\approx 0.37$).

Method 3: Use the Limit Comparison Test with $\sum_{n=0}^\infty \left(\frac{1}{e}\right)^n$ (geometric series with $r=\frac{1}{e}\approx 0.37$).

15. Converges.

Method 1: Use the Direct Comparison Test:

$$\frac{\sqrt{n}}{n^2+1}<\frac{\sqrt{n}}{n^2}=\frac{1}{n^{3/2}}$$

Compare the given series with $\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ (which converges by the p-test).

Method 2: Use the Limit Comparison Test with $\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ (which converges by the p-test).

16. Converges. Use the Limit Comparison Test.

Let $a_n=\frac{5n^3-3n}{n^2(n+2)(n^2+5)}$ and $b_n=\frac{1}{n^2}$.

Then $a_n>0$ and $b_n>0$ for $n\geq 1$, and

$$c=\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{5n^5-3n^3}{n^5+2n^4+5n^3+10n^2}=5$$

Since $0<c<\infty$ and $\sum_{n=1}^\infty \frac{1}{n^2}$ converges by the p-Test,

$\sum_{n=1}^\infty \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$ also converges.

17. Diverges. Use the nth-Term Test:

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{3^{n-1}+1}{3^n}=\lim_{n\to\infty}\frac{1+\frac{1}{3^{n-1}}}{3}=\frac{1}{3}\neq 0$$

Since the sequence of terms does not converge to 0, the series diverges.

18. First, check if the series converges absolutely. (Use the Limit Comparison Test with $\sum_{n=2}^\infty \frac{1}{n}$ to show that $\sum_{n=2}^\infty \frac{1}{\ln n}$ diverges. So, $\sum_{n=2}^\infty (-1)^n\frac{1}{\ln n}$ does not converge absolutely.)

Next, use the Alternating Series Test.

(1) $u_n=\frac{1}{\ln n}>0$ for $n\geq 2$.

(2) We know that $\ln x$ is an increasing function, so $n+1>n\Rightarrow \ln(n+1)>\ln n\Rightarrow \frac{1}{\ln(n+1)}<\frac{1}{\ln n}$. Thus, the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n=\lim_{n\to\infty}\frac{1}{\ln n}=0$

Therefore, $\sum_{n=2}^\infty (-1)^n\frac{1}{\ln n}$ converges.

19. Diverges by the nth-Term Test. Use L'Hôpital's Rule ten times to show that:

$$\lim_{x\to\infty}\frac{10^x}{x^{10}}=\lim_{x\to\infty}\frac{(\ln 10)^{10}10^x}{10!}=\infty$$

This implies that $\lim_{n\to\infty}|a_n|\neq 0$, so $\sum_{n=1}^\infty (-1)^{n+1}\frac{10^n}{n^{10}}$ diverges.

20. Converges by the Alternating Series Test. If $u_n = \frac{\sqrt{n}+1}{n+1}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$. (To show that u_n is decreasing, let

$f(x) = \frac{\sqrt{x}+1}{x+1}$ and observe that

$$f'(x) = \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x}+1)(1)}{(x+1)^2} = \frac{1-x-2\sqrt{x}}{2(x+1)^2\sqrt{x}},$$

which is negative, at least for $x \ge 1$.)

21. Diverges by the nth-Term Test, since $\frac{\ln n}{\ln n^2} = \frac{\ln n}{2\ln n} = \frac{1}{2}$, which means each term is $\pm\frac{1}{2}$.

22. Diverges by the Limit Comparison Test.

Let $a_n = \frac{1}{n} - \frac{1}{n^2}$ and $b_n = \frac{1}{n}$.

Then $a_n > 0$ and $b_n > 0$ for $n \ge 2$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty} a_n$ also diverges.

23. Converges conditionally. First, check for absolute convergence:

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{1}{n^2}\right)$ diverges by the Direct Comparison Test, since $\frac{1}{n} + \frac{1}{n^2} > \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by the p-Test).

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$ does not converge absolutely.

Next, use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{n} + \frac{1}{n^2} > 0$ Clear.

(2) $\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{x^2}\right) = -\frac{1}{x^2} - \frac{2}{x^3} < 0$, for $x > 0$. Thus, the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n^2}\right) = 0$

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$ converges. Truncation error after 99 terms $\le |u_{100}| = 0.0101$

24. Converges absolutely.

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} (0.1)^n$ is a geometric series with $r = 0.1$.

Truncation error after 99 terms $\le |u_{100}| = 10^{-100}$

25. Converges conditionally. First, check for absolute convergence:

$\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges by the Integral Test, since

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{k\to\infty}\left[\ln|\ln x|\right]_2^k = \infty$$

Therefore, $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n\ln n}$ does not converge absolutely.

Next, use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{n\ln n} > 0$ for $n \ge 2$.

(2) $\ln x$ is everywhere increasing, so

$$n+1 > n \Rightarrow \ln(n+1) > \ln n$$
$$\Rightarrow (n+1)\ln(n+1) > n\ln n$$
$$\Rightarrow \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln n}$$

Thus, the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\ln n} = 0$

Therefore, $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n\ln n}$ converges.

Truncation error after 99 terms $\le |u_{100}| = 0.0022$

26. Converges absolutely.

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} n^2\left(\frac{2}{3}\right)^n$ converges by the Ratio Test, since

$$\lim_{n\to\infty} \frac{(n+1)^2(2/3)^{n+1}}{n^2(2/3)^n} = \frac{2}{3}\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^2 = \frac{2}{3} < 1.$$

Truncation error after 99 terms $\le u_{100} = 2.46 \times 10^{-14}$

27. Diverges by the nth-Term Test:

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{n!}{2^n} = \lim_{n\to\infty}\left(\frac{n}{2}\cdot\frac{n-1}{2}\cdot\frac{n-2}{2}\cdots\frac{3}{2}\cdot\frac{2}{2}\cdot\frac{1}{2}\right)$$

$$\geq \lim_{n\to\infty}\left(\frac{n}{2}\cdot 1\cdot 1\cdots 1\cdot 1\cdot\frac{1}{2}\right) = \lim_{n\to\infty}\frac{n}{4} = \infty$$

Since the terms do not converge to 0,

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$ diverges.

28. Converges absolutely.

Since $|a_n| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$, use the Direct Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ (which converges by the p-test).

29. Converges conditionally. First, check for absolute convergence:

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}$ diverges by the Limit Comparison Test (use $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which converges by the p-Test).

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$ does not converge absolutely.

Next, use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{1+\sqrt{n}} > 0$ Clear.

(2) $n+1 > n \Rightarrow \sqrt{n+1} > \sqrt{n}$

$\Rightarrow 1+\sqrt{n+1} > 1+\sqrt{n}$

$\Rightarrow \frac{1}{1+\sqrt{n+1}} < \frac{1}{1+\sqrt{n}}$

Thus the u_n are decreasing.

(3) $\lim_{n\to\infty}u_n = \lim_{n\to\infty}\frac{1}{1+\sqrt{n}} = 0$

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$ converges.

30. Converges absolutely.

$\sum_{n=1}^{\infty}\left|\frac{\cos n\pi}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which converges by the p-Test.

31. Converges conditionally.

$\sum_{n=1}^{\infty}\left|\frac{\cos n\pi}{n}\right| = \sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ (which diverges by the p-Test).

Therefore, $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$ does not converge absolutely.

Next, use the Alternating Series Test to show that

$\sum_{n=1}^{\infty}\frac{\cos n\pi}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges. (See Example 4).

32. Converges conditionally. First, check for absolute convergence:

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges by the Limit Comparison Test. (Let $a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$ and $b_n = \frac{1}{\sqrt{n}}$.

Then

$$c = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}}$$

$$= \lim_{n\to\infty}\frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{2}$$

Since $0 < c < \infty$ and $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (by the p-Test),

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$ also diverges.) Therefore,

$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ does not converge absolutely.

Next, use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} > 0$ Clear.

(2) $\sqrt{n+1}+\sqrt{n+2} > \sqrt{n}+\sqrt{n+1}$

$\Rightarrow \frac{1}{\sqrt{n+1}+\sqrt{n+2}} < \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Thus, the u_n are decreasing.

(3) $\lim_{n\to\infty}u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}} = 0$

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges.

33. The positive terms $2+\frac{4}{3^2}+\frac{6}{5^2}+\cdots+\frac{2n+2}{(2n+1)^2}+\cdots$ diverge to ∞ and the negative terms $-\frac{3}{2^2}-\frac{5}{4^2}-\frac{7}{6^2}-\cdots-\frac{2n+1}{(2n)^2}-\cdots$ diverge to $-\infty$. Answers will vary. Here is one possibility.

(a) Add positive terms until the partial sum is greater than 2. Then add negative terms until the partial sum is less than –2. Then add positive terms until the partial sum is greater than 4. Then add negative terms until the partial sum is less than -4. Repeat this process so that the partial sums swing arbitrarily far in both directions.

(b) Add positive terms until the partial sum is greater than 4. Then add negative terms until the partial sum is less than 4. Continue in this manner indefinitely, always closing in on 4.

34. The positive terms $\frac{1}{3\ln 3}+\frac{1}{5\ln 5}+\frac{1}{7\ln 7}+\cdots+\frac{1}{(2n+1)\ln(2n+1)}+\cdots$ diverge to ∞ and the negative terms $-\frac{1}{2\ln 2}-\frac{1}{4\ln 4}-\frac{1}{6\ln 6}-\cdots-\frac{1}{(2n)\ln(2n)}-\cdots$ diverge to $-\infty$. Answers will vary. Here is one possibility.

(a) Add positive terms until the partial sum is greater than 1. Then add negative terms until the partial sum is less than –1. Then add positive terms until the partial sum is greater than 2. Then add negative terms until the partial sum is less than –2. Repeat this process so that the partial sums swing arbitrarily far in both directions.

(b) Add positive terms until the partial sum is greater than 4. Then add negative terms until the partial sum is less than 4. Continue in this manner indefinitely, always closing in on 4.

35. $\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}|x|^n$, geometric series with $r=|x|$.

The series converges absolutely for $|x|<1$, diverges for $|x|\ge 1$.

(a) Interval of convergence: $(-1, 1)$

(b) Series converges absolutely on $(-1, 1)$

(c) None

36. $\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}|x+5|^n$, geometric series with $r=|x+5|$.

This series converges absolutely for $|x+5|<1$, diverges for $|x+5|\ge 1$.

(a) Interval of convergence: $(-6,-4)$

(b) Series converges absolutely on $(-6,-4)$

(c) None

37. $\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}|4x+1|^n$, geometric series with $r=|4x+1|$.

The series converges absolutely for $|4x+1|<1$, diverges for $|4x+1|\ge 1$.

(a) Interval of convergence: $(-1/2, 0)$

(b) Series converges absolutely on $(-1/2, 0)$

(c) None

38. $\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{|3x-2|^n}{n}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|3x-2|^{n+1}}{n+1}\cdot\frac{n}{|3x-2|^n}$$

$$=|3x-2|\cdot\lim_{n\to\infty}\frac{n}{n+1}=|3x-2|$$

The series converges absolutely for $|3x-2|<1$, or $x\in\left(\frac{1}{3},1\right)$; the series diverges for $|3x-2|>1$

Check $x=1$: $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-Test.

Check $x=\frac{1}{3}$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ does not converge absolutely, but it converges conditionally by the Alternating Series Test.

(a) Interval of convergence: $\left[\frac{1}{3},1\right)$

(b) Series converges absolutely on $\left(\frac{1}{3},1\right)$

(c) Series converges conditionally at $x=\frac{1}{3}$

39. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left|\frac{x-2}{10}\right|^n$, geometric series with $r = \left|\frac{x-2}{10}\right|$.

The series converges absolutely for $\left|\frac{x-2}{10}\right| < 1$, diverges for $\left|\frac{x-2}{10}\right| \geq 1$.

(a) Interval of convergence: (–8, 12)

(b) Series converges absolutely on (–8, 12)

(c) None

40. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n|x|^n}{n+2}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{n+3}\bullet\frac{n+2}{n|x|^n} = |x|\lim_{n\to\infty}\frac{n^2+3n+2}{n^2+3n} = |x|$$

The series converges absolutely for $|x| < 1$, diverges for $|x| > 1$. At $x = \pm 1$, the series diverges by the nth-Term Test.

(a) Interval of convergence: (–1, 1)

(b) Series converges absolutely on (–1, 1)

(c) None

41. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|x|^n}{n\sqrt{n}\,3^n}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)\sqrt{n+1}\bullet 3^{n+1}}\bullet\frac{n\sqrt{n}\cdot 3^n}{|x|^n}$$

$$= \frac{|x|}{3}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2} = \frac{|x|}{3}$$

The series converges absolutely for $|x| < 3$, diverges for $|x| > 3$. When $|x| = 3$, the series also converges absolutely because $\sum_{n=1}^{\infty}\frac{|x|^n}{n\sqrt{n}\,3^n} = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which converges by the p-Test.

(a) Interval of convergence: [–3, 3]

(b) Series converges absolutely on [–3, 3]

(c) None

42. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|x|^{2n+1}}{n!}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|x|^{2n+3}}{(n+1)!}\bullet\frac{n!}{|x|^{2n+1}} = x^2\lim_{n\to\infty}\frac{1}{n+1} = 0$$

The series converges absolutely for all real numbers.

(a) Interval of convergence: $(-\infty, \infty)$

(b) Series converges absolutely for all real numbers

(c) None

43. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n|x+3|^n}{5^n}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)|x+3|^{n+1}}{5^{n+1}}\bullet\frac{5^n}{n|x+3|^n}$$

$$= \frac{|x+3|}{5}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{|x+3|}{5}$$

The series converges absolutely for $|x+3| < 5$ and diverges for $|x+3| > 5$. When $|x+3| = 5$, the series diverges by the nth-Term Test.

(a) Interval of convergence: (–8, 2)

(b) Series converges absolutely on (–8, 2)

(c) None

44. $\sum_{n=1}^{\infty}|a_n| = \frac{n|x|^n}{4^n(n^2+1)}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{4^{n+1}[(n+1)^2+1]}\bullet\frac{4^n(n^2+1)}{n|x|^n}$$

$$= \frac{|x|}{4}\lim_{n\to\infty}\frac{n^3+n^2+n+1}{n^3+2n^2+2n} = \frac{|x|}{4}$$

The series converges absolutely for $|x| < 4$ and diverges for $|x| > 4$.

Check $x = 4$: $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges by the Limit Comparison Test (use $\sum_{n=1}^{\infty}\frac{1}{n}$, which converges by the p-Test).

Check $x = -4$: $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$ does not converge absolutely, but it converges conditionally by the Alternating Series Test.

(a) Interval of convergence: $[-4, 4)$

(b) Series converges absolutely on $(-4, 4)$

(c) Series converges conditionally at $x = -4$

45. $\sum_{n=1}^{\infty}|a_n| = \frac{\sqrt{n}|x|^n}{3^n}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{\sqrt{n+1}|x|^{n+1}}{3^{n+1}} \bullet \frac{3^n}{\sqrt{n}|x|^n} = \frac{|x|}{3}\lim_{n\to\infty}\sqrt{1+\frac{1}{n}} = \frac{|x|}{3}$$

The series converges absolutely for $|x| < 3$ and diverges for $|x| > 3$. When $|x| = 3$, the series diverges by the nth-Term Test.

(a) Interval of convergence: (–3, 3)

(b) Series converges absolutely on (–3, 3)

(c) None

46. $\sum_{n=1}^{\infty}|a_n| = n!|x-4|^n$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)!|x-4|^{n+1}}{n!|x-4|^n}$$
$$= \lim_{n\to\infty}(n+1)|x-4|$$
$$= \begin{cases} 0 & x=4 \\ \infty & x\neq 4 \end{cases}$$

(a) Series only converges at $x = 4$

(b) Series converges absolutely at $x = 4$

(c) None

47. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}2^n(n+1)|x-1|^n$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{2^{n+1}(n+2)|x-1|^{n+1}}{2^n(n+1)|x-1|^n}$$
$$= 2|x-1|\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right) = 2|x-1|$$

The series converges absolutely for $|x-1| < \frac{1}{2}$ and diverges for $|x-1| > \frac{1}{2}$. When $|x-1| = \frac{1}{2}$, the series diverges by the nth-Term Test.

(a) Interval of convergence: $\left(\frac{1}{2}, \frac{3}{2}\right)$

(b) Series converges absolutely on $\left(\frac{1}{2}, \frac{3}{2}\right)$

(c) None

48. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|4x-5|^{2n+1}}{n^{3/2}}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|4x-5|^{2n+3}}{(n+1)^{3/2}} \bullet \frac{n^{3/2}}{|4x-5|^{2n+1}}$$
$$= |4x-5|^2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2} = |4x-5|^2$$

The series converges absolutely for $|4x-5| < 1$, or $x \in \left(1, \frac{3}{2}\right)$; the series diverges for $|4x-5| > 1$.

Check $x = 1$: $\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n^{3/2}} = -\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges absolutely by the p-Test.

Check $x = \frac{3}{2}$: $\sum_{n=1}^{\infty}\frac{1^{2n+1}}{n^{3/2}} = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges absolutely by the p-Test.

(a) Interval of convergence: $\left[1, \frac{3}{2}\right]$

(b) Series converges absolutely on $\left[1, \frac{3}{2}\right]$

(c) None

49. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|x+\pi|^n}{\sqrt{n}}$; use the Ratio Test.

$$\lim_{n\to\infty}\frac{|x+\pi|^{n+1}}{\sqrt{n+1}} \bullet \frac{\sqrt{n}}{|x+\pi|^n}$$
$$= |x+\pi|\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = |x+\pi|$$

The series converges absolutely for $|x+\pi| < 1$, or $x \in (-\pi-1, -\pi+1)$; the series diverges for $|x+\pi| > 1$.

Check $x = -\pi+1$: $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges by the p-Test with $p = \frac{1}{2}$.

Check $x = -\pi-1$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ does not converge absolutely, but it converges conditionally by the Alternating Series Test.

(a) Interval of convergence: $[-\pi-1, -\pi+1)$

(b) Series converges absolutely on $(-\pi-1, -\pi+1)$

(c) Series converges conditionally at $x = -\pi-1$

50. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}|\ln x|^n$, geometric series with $r = |\ln x|$.

The series converges absolutely for $|\ln x| < 1$, or for $e^{-1} < x < e^1$; the series diverges for $|\ln x| \ge 1$.

(a) Interval of convergence: $\left(\frac{1}{e}, e\right)$

(b) Series converges absolutely on $\left(\frac{1}{e}, e\right)$

(c) None

51. $n = 13\times 10^9 \cdot 365 \cdot 24 \cdot 3600 = 4.09968\times 10^{17}$

$\ln(n+1) < \text{sum} < 1 + \ln n$

$\ln(4.09968\times 10^{17} + 1) < \text{sum} < 1 + \ln(4.09968\times 10^{17})$

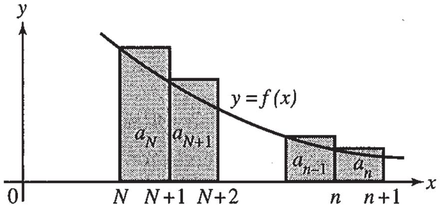
$40.5548\ldots < \text{sum} < 41.5548\ldots$

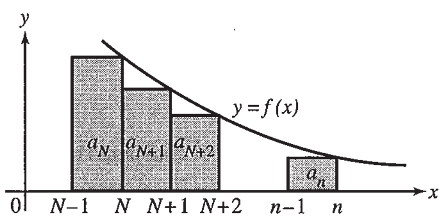
$40.554 < \text{sum} < 41.555$

52. Comparing areas in the figures, we have for all $n \ge 1$, $\int_1^{n+1} f(x)\,dx < a_1 + \cdots + a_n < a_1 + \int_1^n f(x)\,dx$.

If the integral diverges, it must go to infinity, and the first inequality forces the partial sums of the series to go to infinity as well, so the series is divergent. If the integral converges, then the second inequality puts an upper bound on the partial sums of the series, and since they are a nondecreasing sequence, they must converge to a finite sum for the series. (See the explanation preceding Exercises 61 and 62 in Section 9.4.)

53.





Comparing areas in the figures, we have for all $n \ge N$, $\int_N^{n+1} f(x)\,dx < a_N + \cdots + a_n < a_N + \int_N^n f(x)\,dx$.

If the integral diverges, it must go to infinity, and the first inequality forces the partial sums of the series to go to infinity as well, so the series is divergent. If the integral converges, then the second inequality puts an upper bound on the partial sums of the series, and since they are a nondecreasing sequence, they must converge to a finite sum for the series. (See the explanation preceding Exercises 61 and 62 in Section 9.4.)

54. (a) Diverges by the Limit Comparison Test.

Let $a_k = \dfrac{1}{\sqrt{2k+7}}$ and $b_k = \dfrac{1}{k^{1/2}}$. Then $a_k > 0$ and $b_k > 0$ for $k \ge 1$ and $\lim_{k\to\infty}\dfrac{a_k}{b_k} = \lim_{k\to\infty}\dfrac{k^{1/2}}{\sqrt{2k+7}} = \dfrac{1}{\sqrt{2}}$.

Since $\sum_{k=1}^{\infty} b_k$ diverges by the p-Test with $p = \frac{1}{2}$, $\sum_{k=1}^{\infty} a_k$ also diverges.

(b) Diverges by the nth-Term Test, since

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e \ne 0.$$

(c) Converges absolutely by the Direct Comparison Test, since $\left|\dfrac{\cos k}{k^2+\sqrt{k}}\right| < \dfrac{1}{k^2}$ for $k \ge 1$ and $\sum_{k=1}^{\infty}\dfrac{1}{k^2}$ converges by the p-Test with $p = 2$.

(d) Diverges by the Integral Test, since

$$\int_3^{\infty}\frac{18}{x\ln x}\,dx = \lim_{b\to\infty}\left[18\ln|\ln x|\right]_3^b = \infty.$$

55. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\dfrac{n^n|x+2|^n}{3^n n!}$, use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)^{n+1}|x+2|^{n+1}}{3^{n+1}(n+1)!}\cdot\frac{3^n n!}{n^n|x+2|^n}$$
$$= \frac{|x+2|}{3}\lim_{n\to\infty}\frac{(n+1)^n}{n^n}$$
$$= \frac{|x+2|}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$
$$= \frac{|x+2|}{3}\cdot e = |x+2|\cdot\frac{e}{3}$$

The series converges absolutely for $|x+2| < \dfrac{3}{e}$ and diverges for $|x+2| > \dfrac{3}{e}$. Therefore, the radius of convergence is $3/e$.

56. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n!|x|^n}{n^n 5^n}$, use the Ratio Test.

$$\lim_{n\to\infty}\frac{(n+1)!|x|^{n+1}}{(n+1)^{n+1}5^{n+1}}\bullet\frac{n^n 5^n}{n!|x|^n}$$
$$=\frac{|x|}{5}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$
$$=\frac{|x|}{5}\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{|x|}{5e}$$

The series converges absolutely for $|x| < 5e$, diverges for $|x| > 5e$. Therefore, the radius of convergence is $5e$.

57. One possible answer: $\sum_{n=3}^{\infty}\frac{1}{n\ln n}$

This series diverges by the integral test, since $\int_3^{\infty}\frac{1}{x\ln x}dx = \lim_{b\to\infty}\left[\ln|\ln x|\right]_3^b = \infty$. Its partial sums are roughly ln(ln n), so they are much smaller than the partial sums for the harmonic series, which are about ln n.

58. (a) $a_k = (-1)^{k+1}\int_0^{1/k}6(kx)^2\,dx$
$$=(-1)^{k+1}\left[2k^2x^3\right]_0^{1/k}$$
$$=(-1)^{k+1}\left(\frac{2}{k}\right)$$

(b) The series converges by the Alternating Series Test.

(c) The first few partial sums are:

$S_1 = 2, S_2 = 1, S_3 = \frac{5}{3}, S_4 = \frac{7}{6}, S_5 = \frac{47}{30}, S_6 = \frac{37}{30}$,

$S_7 = \frac{319}{210}, S_8 = \frac{533}{420}, S_9 = \frac{1879}{1260}$. For an alternating series, the sum is between any two adjacent partial sums, so

$1 < S_8 \le \text{sum} \le S_9 < \frac{3}{2}$.

59. (a) Diverges by the Limit Comparison Test. Let

$a_n = \frac{n}{3n^2+1}$ and $b_n = \frac{1}{n}$. Then $a_n > 0$ and $b_n > 0$ for $n \ge 1$, and $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{3n^2+1} = \frac{1}{3}$. Since $\sum_{n=1}^{\infty}b_n$ diverges, $\sum_{n=1}^{\infty}a_n$ diverges.

(b) $S = \sum_{n=1}^{\infty}\frac{n}{3n^2+1}\bullet\frac{3}{n} = \sum_{n=1}^{\infty}\frac{3}{3n^2+1}$.

This series converges by the Direct Comparison Test, since $\frac{3}{3n^2+1} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-Test with $p = 2$.

60. (a) From the list of Maclaurin series in Section 9.2,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

(b) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}}{n+1}\bullet\frac{n}{|x|^n} = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|$

The series converges for $|x| < 1$ and diverges for $|x| > 1$.

Check $x = 1$: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges by the Alternating Series Test.

Check $x = -1$: $-\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-Test.

The series converges for $-1 < x \le 1$.

(c) To estimate $\ln\frac{3}{2}$, we would let $x = \frac{1}{2}$.

The truncation error is less than the magnitude of the sixth nonzero term, or

$$\left|-\frac{x^6}{6}\right| = \frac{1}{2^6\bullet 6} = \frac{1}{384} < 0.002605.$$

Thus, a bound for the (absolute) truncation error is 0.002605.

(d) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x^2)^n}{n} = \frac{1}{2}\ln(1+x^2)$

61. $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{2^{k+1}|x|^{k+1}}{\ln(k+3)}\bullet\frac{\ln(k+2)}{2^k|x|^k} = 2|x|$

The series converges absolutely for $|x| < \frac{1}{2}$, or $-\frac{1}{2} < x < \frac{1}{2}$.

Check $x = -\frac{1}{2}$:

$\sum_{k=0}^{\infty}\frac{(-1)^k}{\ln(k+2)}$ converges by the Alternating Series Test.

61. Continued

Check $x = \frac{1}{2}$:

$\sum_{k=0}^{\infty} \frac{1}{\ln(k+2)}$ diverges by the Direct Comparison Test,

since $\frac{1}{\ln(k+2)} > \frac{1}{k}$ for all $k \ge 2$, and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (harmonic).

The original series converges for $-\frac{1}{2} \le x < \frac{1}{2}$.

62. (a) The series converges by the Direct Comparison Test,

since $\frac{1}{n^p \ln n} < \frac{1}{n^p}$ for $n \ge 3$, and $\sum_{n=3}^{\infty} \frac{1}{n^p}$ converges as a p-series when $p > 1$.

(b) For $p = 1$, the series is $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$, which diverges by the Integral Test, since

$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \left[\ln(\ln x)\right]_2^b = \infty.$

(c) For $0 \le p < 1$, we have $\frac{1}{n^p \ln n} > \frac{1}{n \ln n}$, so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges by the Direct Comparison Test with $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ from part (b).

63. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, so at $x = 1$, the series is

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. This series converges by the Alternating Series Test.

64. $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

At $x = -1$, the series is

$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, which converges by the Alternating Series Test. At $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, which converges by the Alternating Series Test.

65. (a) It fails to satisfy $u_n \ge u_{n+1}$ for all $n \ge N$.

(b) The sum is $\left(\sum_{n=1}^{\infty} \frac{1}{3^n}\right) - \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) = \frac{1/3}{1-1/3} - \frac{1/2}{1-1/2}$

$$= \frac{1}{2} - 1$$

$$= -\frac{1}{2}.$$

66. True, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|x|^{2n}}{2n}$

Use the Ratio Test to find the endpoints.

$$\lim_{n \to \infty} \frac{|x|^{2(n+1)}}{2(n+1)} \cdot \frac{2n}{|x|^{2n}} = |x|^2 \lim_{n \to \infty} \frac{n}{n+1} = |x|^2$$

The series converges absolutely for $|x|^2 < 1$, or $|x| < 1$. The endpoints are $x = \pm 1$. At both endpoints the series equals: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$, which converges by the Alternating Series Test.

67. True. The next term is $a_{101} = \frac{(-1)^{101}}{101^2}$, which is negative, so s_{100} must be greater than the sum of the series.

68. B. $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n|2x-5|^n}{n+2}$, use the Ratio Test.

$$\lim_{n \to \infty} \frac{(n+1)|2x-5|^{n+1}}{n+3} \cdot \frac{n+2}{n|2x-5|^n}$$

$$= |2x-5| \lim_{n \to \infty} \frac{n^2+3n+2}{n^2+3n} = |2x-5|$$

The series converges absolutely when $|2x-5| < 1$, or $\left|x - \frac{5}{2}\right| < \frac{1}{2}$.

69. A. The series converges absolutely for $\left|x - \frac{5}{2}\right| < \frac{1}{2}$; the series diverges at the endpoints by the nth-Term Test. The interval of convergence is: $2 < x < 3$.

70. E.

I. $4\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-Test.

II. $\sum_{n=1}^{\infty} \left(\frac{1}{\ln 4}\right)^n$ converges (geometric series with $r \approx 0.7$).

III. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely by the p-Test (or use the Alternating Series Test).

71. B. The truncation error is $\left|\sum_{n=101}^{\infty}\frac{(-1)^n}{2^n}\right| = \left|\sum_{n=101}^{\infty}\left(-\frac{1}{2}\right)^n\right|.$

Notice that $\sum_{n=101}^{\infty}\left(-\frac{1}{2}\right)^n$ is a geometric series with first term $\left(-\frac{1}{2}\right)^{101}$ and constant ratio $r=-\frac{1}{2}$. It converges to

$$\frac{\left(-\frac{1}{2}\right)^{101}}{1-\left(-\frac{1}{2}\right)} = -\frac{\left(\frac{1}{2}\right)^{101}}{\frac{3}{2}} = -\frac{1}{3\cdot 2^{100}}.$$ Thus $\frac{1}{3\cdot 2^{100}}$ is the truncation error.

72. Answers will vary.

73. (a) $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty}\frac{(\sqrt[n]{n})^2}{2} = \frac{1}{2}$ The series converges.

(b) $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n-1}\right)^n} = \lim_{n\to\infty}\frac{n}{2n-1} = \frac{1}{2}$ The series converges.

(c) $$\lim_{n\to\infty,\, n \text{ odd}}\sqrt[n]{a_n} = \lim_{n\to\infty,\, n\text{ odd}}\sqrt[n]{\frac{n}{2^n}} = \lim_{n\to\infty,\, n\text{ odd}}\frac{\sqrt[n]{n}}{2} = \frac{1}{2}$$

$$\lim_{n\to\infty,\, n\text{ even}}\sqrt[n]{a_n} = \lim_{n\to\infty,\, n\text{ even}}\sqrt[n]{\frac{1}{2^n}} = \frac{1}{2}$$

Thus, $\lim_{n\to\infty}\sqrt[n]{a_n} = \frac{1}{2}$, so the series converges.

74. (a) $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{|x-1|^n}{4n}} = \frac{|x-1|}{4}.$

The series converges absolutely if

$\frac{|x-1|}{4} < 1$, or $-3 < x < 5$.

Check $x=-3$: $\sum_{n=0}^{\infty}(-1)^n$ diverges.

Check $x=5$: $\sum_{n=0}^{\infty}1^n$ diverges.

The interval of convergence is $(-3, 5)$.

(b) $$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{|x-2|^n}{n\cdot 3^n}} = \lim_{n\to\infty}\frac{|x-2|}{\sqrt[n]{n}\cdot 3} = \frac{|x-2|}{3}.$$

The series converges absolutely if

$\frac{|x-2|}{3} < 1$, or $-1 < x < 5$.

Check $x=-1$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges.

Check $x=5$: $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

The interval of convergence is $[-1, 5)$.

(c) $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{2^n|x|^n} = 2|x|$. The series converges absolutely if $2|x| < 1$, or $-\frac{1}{2} < x < \frac{1}{2}$.

Check $x=-\frac{1}{2}$: $\sum_{n=1}^{\infty}(-1)^n$ diverges.

Check $x=\frac{1}{2}$: $\sum_{n=1}^{\infty}1$ diverges.

The interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

(d) $$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{|\ln x|^n} = |\ln x|.$$

The series converges absolutely if

$|\ln x| < 1$, or $\frac{1}{e} < x < e$.

Check: $x=\frac{1}{e}$: $\sum_{n=0}^{\infty}\left(\ln\frac{1}{e}\right)^n = \sum_{n=0}^{\infty}(-1)^n$ diverges.

Check $x=e$: $\sum_{n=0}^{\infty}(\ln e)^n = \sum_{n=0}^{\infty}1^n$ diverges.

The interval of convergence is $\left(\frac{1}{e}, e\right)$.