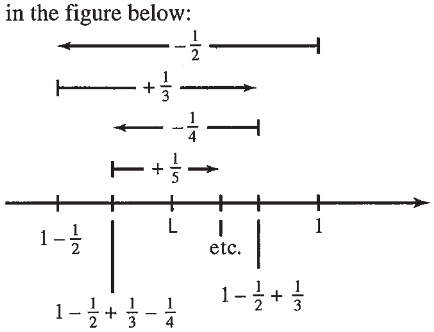
3. Geometrically, we chart the progress of the partial sums as in the figure below:



4. The series converges at the right-hand endpoint. As shown in the picture above, the partial sums are closing in on some limit L as they oscillate left and right by constantly decreasing amounts.

5. We know that the series does not converge absolutely at the right-hand endpoint, because $\sum \frac{1}{n}$ diverges (Exploration 1 of this section).

Quick Review 9.4

1. $\lim_{n\to\infty} \frac{n|x|}{n+1} = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|$

2. $\lim_{n\to\infty} \frac{n^2|x-3|}{n(n-1)} = |x-3| \cdot \lim_{n\to\infty} \frac{n^2}{n^2-n}$

$= |x-3| \cdot \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}} = |x-3|$

3. $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$

(Note: This limit is similar to the limit which is discussed at the end of Example 3 in Section 9.3.)

4. $\lim_{n\to\infty} \frac{(n+1)^4 x^2}{(2n)^4} = x^2 \lim_{n\to\infty} \frac{n^4+4n^3+6n^2+4n+1}{16n^4}$

$= x^2\left(\frac{1}{16}\right) = \frac{x^2}{16}$

5. $\lim_{n\to\infty} \frac{|2x+1|^{n+1} 2^n}{2^{n+1}|2x+1|^n} = \lim_{n\to\infty} \frac{|2x+1|}{2} = \frac{|2x+1|}{2}$

6. Since $n^2 > 5n$ implies $n > 5$,

take $a_n = n^2, b_n = 5n$ and $N = 6$.

Then $a_n > b_n$ for all $n \ge N$.

7. Clearly, $a_5 = b_5$. (They both equal 5^5.)

Using a calculator to graph the functions x^5 and 5^x, we see that $5^x > x^5$ for $x > 5$.

Take $a_n = 5^n, b_n = n^5$ and $N = 6$.

Then $a_n > b_n$ for all $n \ge N$.

8. We showed in Section 8.3, Exercise 6 that $\sqrt{x}$ grows faster than $\ln x$. This and the fact that $\sqrt{1} > \ln 1$ tell us that $\sqrt{x} > \ln x$ for all $x \ge 1$ ($\ln x$ starts behind and will never catch up).

Take $a_n = \sqrt{n}, b_n = \ln n$ and $N = 1$. Then $a_n > b_n$ for all $n \ge N$.

9. We showed in Exercise 3 that for any x, $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$.

Rewriting this limit:

$$0 = \lim_{n\to\infty} \frac{10^n}{n!} = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{10^n}}$$

we see that $\frac{1}{10^n}$ grows faster than $\frac{1}{n!}$.

Take $a_n = \frac{1}{10^n}$ and $b_n = \frac{1}{n!}$

Evaluate a_n and b_n, for $n = 0, 1, 2, 3, \ldots$

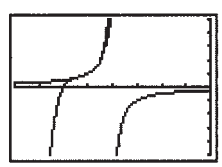
We see that 25 is the first value of n for which $a_n > b_n$.

Take $N = 25$. Since $a_{25} > b_{25}$, and a_n grows faster than b_n, it follows that $a_n > b_n$ for all $n \ge N$.

10. Since $n^2 < n^3$ for all $n > 1$, it follows that $\frac{1}{n^2} > \frac{1}{n^3} = n^{-3}$ for $n \ge 2$. Take $a_n = \frac{1}{n^2}, b_n = n^{-3}$ and $N = 2$. Then $a_n \ge b_n$ for all $n \ge N$.

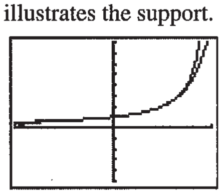
Section 9.4 Exercises

1. $-6 < x < -4$; The graph of the function $y = \frac{-1}{x+4}$ and $P_5(x)$ illustrates the support.



[–3, 3] by [–2, 2]

2. $-1 < x < 1$; The graph of the function $y = \frac{1}{1-x}$ and $P_9(x)$ illustrates the support.



[–1, 1] by [–5, 8]

3. $\dfrac{x^{3n}}{2n!+1} \le \dfrac{x^{3n}}{n!} \le \dfrac{(x^3)^n}{n!}$ and $\sum_{n=0}^{\infty} \dfrac{(x^3)^n}{n!}$ is the Taylor series for e^{x^3} which converges for all x.

4. $\dfrac{x^{2n}}{n!+2} \le \dfrac{x^{2n}}{n!} \le \dfrac{(x^2)^n}{n!}$ and $\sum_{n=0}^{\infty} \dfrac{(x^2)^n}{n!}$ is the Taylor series for e^{x^2} which converges for all x.

5. $\left|\dfrac{(\cos x)^n}{n!+1}\right| \le \dfrac{|\cos x|^n}{n!} \le \dfrac{1}{n!}$ and $\sum_{n=0}^{\infty} \dfrac{1}{n!}$ converges to e.

6. $\left|\dfrac{2(\sin x)^n}{n!+3}\right| \le \dfrac{2|\sin x|^n}{n!} \le \dfrac{2}{n!}$ and $\sum_{n=0}^{\infty} \dfrac{2}{n!}$ converges to $2e$.

7. This is a geometric series which converges only for $|x| < 1$, so the radius of convergence is 1.

8. This is a geometric series which converges only for $|x+5| < 1$, so the radius of convergence is 1.

9. This is a geometric series which converges only for $|-(4x+1)| < 1$, or $\left|x+\dfrac{1}{4}\right| < \dfrac{1}{4}$, so the radius of convergence is $\dfrac{1}{4}$.

10. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{|3x-2|^{n+1}}{n+1} \bullet \dfrac{n}{|3x-2|^n} = |3x-2|$

The series converges for $|3x-2| < 1$, or $\left|x-\dfrac{2}{3}\right| < \dfrac{1}{3}$, and diverges for $\left|x-\dfrac{2}{3}\right| > \dfrac{1}{3}$, so the radius of convergence is $\dfrac{1}{3}$.

11. This is a geometric series which converges only for $\left|\dfrac{x-2}{10}\right| < 1$, or $|x-2| < 10$, so the radius of convergence is 10.

12.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{n+3} \bullet \frac{n+2}{n|x|^n} = |x| \bullet \lim_{n\to\infty} \frac{n^2+3n+2}{n^2+3n} = |x|$$

The series converges for $|x| < 1$ and diverges for $|x| > 1$, so the radius of convergence is 1.

13.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)\sqrt{n+1}\,3^{n+1}} \bullet \frac{n\sqrt{n}\,3^n}{|x|^n} = \frac{|x|}{3} \bullet \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2} = \frac{|x|}{3}$$

The series converges for $|x| < 3$ and diverges for $|x| > 3$, so the radius of convergence is 3.

14. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{|x|^{2n+3}}{(n+1)!} \bullet \dfrac{n!}{|x|^{2n+1}} = \lim_{n\to\infty} \dfrac{x^2}{n+1} = 0$

The series converges for all values of x, so the radius of convergence is ∞.

15.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)|x+3|^{n+1}}{5^{n+1}} \bullet \frac{5^n}{n|x+3|^n} = \frac{|x+3|}{5} \bullet \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x+3|}{5}$$

The series converges for $|x+3| < 5$ and diverges for $|x+3| > 5$, so the radius of convergence is 5.

16.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{4^{n+1}[(n+1)^2+1]} \bullet \frac{4^n(n^2+1)}{n|x|^n} = \frac{|x|}{4} \bullet \lim_{n\to\infty} \frac{n^3+n^2+n+1}{n^3+2n^2+2n} = \frac{|x|}{4}$$

The series converges for $|x| < 4$ and diverges for $|x| > 4$, so the radius of convergence is 4.

17.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!|x-4|^{n+1}}{n!|x-4|^n} = \lim_{n\to\infty} (n+1)|x-4| = \infty \ (x \ne 4)$$

The series converges only for $x = 4$, so the radius of convergence is 0.

18.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n+1}|x|^{n+1}}{3^{n+1}} \bullet \frac{3^n}{\sqrt{n}|x|^n} = \frac{|x|}{3} \bullet \lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = \frac{|x|}{3}$$

The series converges for $|x| < 3$ and diverges for $|x| > 3$, so the radius of convergence is 3.

19.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|(-2)^{n+1}|(n+2)|x-1|^{n+1}}{|(-2)^n|(n+1)|x-1|^n} = 2|x-1| \bullet \lim_{n\to\infty} \frac{n+2}{n+1} = 2|x-1|$$

The series converges for $|x-1| < \dfrac{1}{2}$ and diverges for $|x-1| > \dfrac{1}{2}$, so the radius of convergence is $\dfrac{1}{2}$.

20. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|4x-5|^{2n+3}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{|4x-5|^{2n+1}}$

$= (4x-5)^2 \cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2}$

$= (4x-5)^2$

The series converges for $(4x-5)^2 < 1$, which is equivalent to $|4x-5| < 1$, or $\left|x-\frac{5}{4}\right| < \frac{1}{4}$ and diverges for $\left|x-\frac{5}{4}\right| > \frac{1}{4}$.

The radius of convergence is $\frac{1}{4}$.

21. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x+\pi|^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{|x+\pi|^n}$

$= |x+\pi| \cdot \lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = |x+\pi|$

The series converges for $|x+\pi| < 1$ and diverges for $|x+\pi| > 1$, so the radius of convergence is 1.

22. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left|x-\sqrt{2}\right|^{2n+3}}{2^{n+1}}\cdot\frac{2^n}{\left|x-\sqrt{2}\right|^{2n+1}}$

$= \lim_{n\to\infty}\frac{1}{2}(x-\sqrt{2})^2 = \frac{1}{2}(x-\sqrt{2})^2$

The series converges for $\frac{1}{2}(x-\sqrt{2})^2 < 1$, which is equivalent to $\left|x-\sqrt{2}\right| < \sqrt{2}$, and diverges for $\left|x-\sqrt{2}\right| > \sqrt{2}$. The radius of convergence is $\sqrt{2}$.

23. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{(x-1)^2}{4}$. It converges only when $\left|\frac{(x-1)^2}{4}\right| < 1$, so the interval of convergence is $-1 < x < 3$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{(x-1)^2}{4}} = \frac{4}{4-(x-1)^2} = \frac{4}{-x^2+2x+3} = -\frac{4}{x^2-2x-3}$$

24. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{(x+1)^2}{9}$. It converges only when $\left|\frac{(x+1)^2}{9}\right| < 1$, so the interval of convergence is $-4 < x < 2$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{(x+1)^2}{9}} = \frac{9}{9-(x+1)^2} = \frac{9}{-x^2-2x+8} = -\frac{9}{x^2+2x-8}$$

25. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{\sqrt{x}}{2} - 1$. It converges only when $\left|\frac{\sqrt{x}}{2} - 1\right| < 1$, so the interval of convergence is $0 < x < 16$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\left(\frac{\sqrt{x}}{2}-1\right)} = \frac{2}{4-\sqrt{x}}$$

26. This is a geometric series with first term $a = 1$ and common ratio $r = \ln x$. It converges only when $|\ln x| < 1$, so the interval of convergence is $\frac{1}{e} < x < e$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\ln x}$$

27. This is a geometric series with first term $a = 1$ and common ratio $\frac{x^2-1}{3}$. It converges only when $\left|\frac{x^2-1}{3}\right| < 1$, which is equivalent to $-2 < x^2 < 4$. It is always true that $x^2 > -2$, and $x^2 < 4$ implies that $|x| < 2$, so the interval of convergence is $-2 < x < 2$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{x^2-1}{3}} = \frac{3}{3-(x^2-1)} = \frac{3}{4-x^2}$$

28. This is a geometric series with first term $a = 1$ and common ratio $\frac{\sin x}{2}$. Since $\left|\frac{\sin x}{2}\right| < 1$ for all x, the interval of convergence is $-\infty < x < \infty$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{\sin x}{2}} = \frac{2}{2-\sin x}$$

29. Diverges by the nth-Term Test, since $\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$.

30. Diverges by the nth-Term Test, since $\lim_{n\to\infty}\frac{2^n}{n+1} = \infty$. (The Ratio Test can also be used.)

31. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^2-1}{2^{n+1}}\cdot\frac{2^n}{n^2-1} = \frac{1}{2} < 1.$$

32. Converges, because it is a geometric series with $r = \frac{1}{8}$, so $|r| < 1$.

33. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{n+1}}{(3^{n+1}+1)}\bullet\frac{3^n+1}{2^n}=\frac{2}{3}<1.$$

Alternate method: Note that $\dfrac{2^n}{3^n+1}<\left(\dfrac{2}{3}\right)^n$ for all n. Since

$\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges, $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{3^n+1}$ converges by the Direct Comparison Test.

34. Diverges by the nth-Term Test, since

$$\begin{aligned}\lim_{n\to\infty}n\sin\left(\frac{1}{n}\right)&=\lim_{n\to\infty}\frac{\sin(1/n)}{(1/n)}\\&=\lim_{x\to 0}\frac{\sin x}{x}\qquad\text{(where } x=1/n\text{)}\\&=1\neq 0\end{aligned}$$

35. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^2e^{-n-1}}{n^2e^{-n}}=e^{-1}<1.$$

36. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^{10}}{10^{n+1}}\bullet\frac{10^n}{n^{10}}=\frac{1}{10}<1.$$

37. Converges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}&=\lim_{n\to\infty}\frac{(n+4)!}{3!(n+1)!3^{n+1}}\bullet\frac{3!n!3^n}{(n+3)!}\\&=\lim_{n\to\infty}\frac{n+4}{3(n+1)}\\&=\frac{1}{3}<1.\end{aligned}$$

38. Diverges by the nth-Term Test, since

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\neq 0.$$

39. Converges, because it is a geometric series with $r=-\dfrac{2}{3}$, so $|r|<1$.

40. Diverges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!e^{-n-1}}{n!e^{-n}}=\lim_{n\to\infty}(n+1)e^{-1}=\infty.$$

(The nth-Term Test can also be used.)

41. Diverges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}&=\lim_{n\to\infty}\frac{3^{n+1}}{(n+1)^3 2^{n+1}}\bullet\frac{n^3 2^n}{3^n}\\&=\lim_{n\to\infty}\frac{3n^3}{(n+1)^3(2)}\\&=\frac{3}{2}>1.\end{aligned}$$

(The nth-Term Test can also be used.)

42. Converges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}&=\lim_{n\to\infty}\frac{(n+1)\ln(n+1)}{2^{n+1}}\bullet\frac{2^n}{n\ln n}\\&=\lim_{n\to\infty}\frac{1}{2}\bullet\frac{n+1}{n}\bullet\frac{\ln(n+1)}{\ln n}\\&=\lim_{n\to\infty}\frac{1}{2}\bullet\frac{1}{1}\bullet\frac{1/(n+1)}{1/n}\qquad\text{(L'Hôpital's Rule)}\\&=\frac{1}{2}\bullet\lim_{n\to\infty}\bullet\frac{n}{n+1}=\frac{1}{2}<1\end{aligned}$$

43. Converges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}&=\lim_{n\to\infty}\frac{(n+1)!}{(2n+3)!}\bullet\frac{(2n+1)!}{n!}\\&=\lim_{n\to\infty}\frac{n+1}{(2n+3)(2n+2)}\\&=\lim_{n\to\infty}\frac{n+1}{4n^2+10n+2}\\&=\lim_{n\to\infty}\frac{1}{8n+10}\qquad\text{(L'Hôpital's Rule)}\\&=0<1\end{aligned}$$

44. Converges by the Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}&=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\bullet\frac{n^n}{n!}=\lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)^{n+1}}\\&=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\frac{1}{(1+1/n)^n}\\&=\frac{1}{e}<1\end{aligned}$$

45. One possible answer: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (see Exploration 1 in this section) even though $\displaystyle\lim_{n\to\infty}\frac{1}{n}=0$.

46. One possible answer:

Let $a_n=2^{-n}$ and $b_n=3^{-n}$

Then $\sum a_n$ and $\sum b_n$ are convergent geometric series, but

$\displaystyle\sum\frac{a_n}{b_n}=\sum\left(\frac{3}{2}\right)^n$ is a divergent geometric series.

47. Almost, but the Ratio Test won't determine whether there is convergence or divergence at the endpoints of the interval.

48. $\displaystyle\sum_{n=1}^{\infty}\frac{4}{(4n-3)(4n+1)}=\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)$

$$\begin{aligned}s_1&=1-\frac{1}{5}\\s_2&=\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)=1-\frac{1}{9}\\s_3&=\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)=1-\frac{1}{13}\\s_n&=1-\frac{1}{4n+1}\\S&=\lim_{n\to\infty}s_n=1\end{aligned}$$

49. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty}\left(\frac{3}{2n-1} - \frac{3}{2n+1}\right)$

$s_1 = 3 - \frac{3}{3}$

$s_2 = (3-1) + \left(1 - \frac{3}{5}\right) = 3 - \frac{3}{5}$

$s_3 = (3-1) + \left(1 - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{7}\right) = 3 - \frac{3}{7}$

$s_n = 3 - \frac{3}{2n+1}$

$S = \lim_{n\to\infty} s_n = 3$

50. $\frac{40n}{(2n-1)^2(2n+1)^2} = \frac{A}{(2n-1)^2} + \frac{B}{(2n+1)^2}$

$A(2n+1)^2 + B(2n-1)^2 = 40n$

$n = -\frac{1}{2} \Rightarrow 4B = -20 \Rightarrow B = -5$

$n = \frac{1}{2} \Rightarrow 4B = 20 \Rightarrow A = 5$

$\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2} = \sum_{n=1}^{\infty}\left[\frac{5}{(2n-1)^2} - \frac{5}{(2n+1)^2}\right]$

$s_1 = 5 - \frac{5}{9}$

$s_2 = \left(5 - \frac{5}{9}\right) + \left(\frac{5}{9} - \frac{5}{25}\right) = 5 - \frac{5}{25}$

$s_3 = \left(5 - \frac{5}{9}\right) + \left(\frac{5}{9} - \frac{5}{25}\right) + \left(\frac{5}{25} - \frac{5}{49}\right) = 5 - \frac{5}{49}$

$s_n = 5 - \frac{5}{(2n+1)^2}$

$S = \lim_{n\to\infty} s_n = 5$

51. $\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n^2} + \frac{B}{(n+1)^2}$

$A(n+1)^2 + Bn^2 = 2n+1$

$n = 0 \Rightarrow A = 1$

$n = -1 \Rightarrow B = -1$

$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$

$s_1 = 1 - \frac{1}{4}$

$s_2 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) = 1 - \frac{1}{9}$

$s_3 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{16}\right) = 1 - \frac{1}{16}$

$s_n = 1 - \frac{1}{(n+1)^2}$

$S = \lim_{n\to\infty} s_n = 1$

52. $s_1 = 1 - \frac{1}{\sqrt{2}}$

$s_2 = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) = 1 - \frac{1}{\sqrt{3}}$

$s_3 = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) = 1 - \frac{1}{\sqrt{4}}$

$s_n = 1 - \frac{1}{\sqrt{n+1}}$

$S = \lim_{n\to\infty} s_n = 1$

53. $s_1 = \frac{1}{\ln 3} - \frac{1}{\ln 2}$

$s_2 = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) = \frac{1}{\ln 4} - \frac{1}{\ln 2}$

$s_3 = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 5} - \frac{1}{\ln 4}\right)$

$= \frac{1}{\ln 5} - \frac{1}{\ln 2}$

$s_n = \frac{1}{\ln(n+2)} - \frac{1}{\ln 2}$

$S = \lim_{n\to\infty} s_n = -\frac{1}{\ln 2}$

54. $s_1 = \tan^{-1} 1 - \tan^{-1} 2 = \frac{\pi}{4} - \tan^{-1} 2$

$s_2 = (\tan^{-1} 1 - \tan^{-1} 2) + (\tan^{-1} 2 - \tan^{-1} 3)$

$= \frac{\pi}{4} - \tan^{-1} 3$

$s_3 = (\tan^{-1} 1 - \tan^{-1} 2) + (\tan^{-1} 2 - \tan^{-1} 3)$
$+ (\tan^{-1} 3 - \tan^{-1} 4)$

$= \frac{\pi}{4} - \tan^{-1} 4$

$s_n = \frac{\pi}{4} - \tan^{-1}(n+1)$

$S = \lim_{n\to\infty} s_n = \frac{\pi}{4} - \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

55. True. See Theorem 8.

56. False. The power series

$\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges at $x = a$. (The sum of the series is c_0.)

57. B. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(-3)^{n+1}} \bullet \frac{(-3)^n}{2^n} = \lim_{n\to\infty} \frac{2}{-3} = -\frac{2}{3}$

58. C. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|2x-3|^{n+1}}{n+1} \bullet \frac{n}{|2x-3|^n}$

$= |2x-3| \bullet \lim_{n\to\infty} \frac{n}{n+1}$

$= |2x-3|$

The series converges for $|2x-3| < 1$, which is equivalent to $\left|x - \frac{3}{2}\right| < \frac{1}{2}$, and diverges for $\left|x - \frac{3}{2}\right| > \frac{1}{2}$.

The radius of convergence is 1/2.

59. E. Consider first the series $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|\sin x|^n}{2^n n^2}$

For any real number x, and for all n:

$$\frac{|\sin x|^n}{2^n n^2} \le \frac{1}{2^n n^2} < \frac{1}{2^n}$$

Since $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges, $\sum_{n=1}^{\infty}\frac{|\sin x|^n}{2^n n^2}$ converges by the Direct Comparison Test.

Then, by Theorem 8, since $\sum_{n=1}^{\infty}\frac{(\sin x)^n}{2^n n^2}$ converges absolutely, it also converges.

60. D. $\sum_{n=1}^{\infty}\frac{3}{(3n-1)(3n+2)} = \sum_{n=1}^{\infty}\frac{1}{3n-1} - \frac{1}{3n+2}$

$$s_1 = \frac{1}{2} - \frac{1}{5}$$

$$s_2 = \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) = \frac{1}{2} - \frac{1}{8}$$

$$s_3 = \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{11}\right) = \frac{1}{2} - \frac{1}{11}$$

$$s_n = \frac{1}{2} - \frac{1}{3n+2}$$

$$S = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{2}$$

61. **(a)** For $k \le N$, it's obvious that

$$S_k = a_1 + \cdots + a_k \le a_1 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n.$$

For all $k > N$,

$$\begin{aligned} S_k &= a_1 + \cdots + a_k = a_1 + \cdots + a_N + a_{N+1} + \cdots + a_k \\ &\le a_1 + \cdots + a_N + c_{N+1} + \cdots + c_k \\ &\le a_1 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n \end{aligned}$$

(b) Since all of the a_n are nonnegative, the partial sums of the series form a nondecreasing sequence of real numbers. Part (a) shows that this sequence is bounded above, so it must converge to a limit.

62. **(a)** For $k \le N$, it's obvious that

$$d_1 + \cdots + d_k \le d_1 + \cdots + d_N + \sum_{n=N+1}^{\infty} a_n.$$

For all $k > N$,

$$\begin{aligned} &d_1 + \cdots + d_k = d_1 + \cdots + d_N + d_{N+1} + \cdots + d_k \\ &\le d_1 + \cdots + d_N + a_{N+1} + \cdots + a_k \\ &\le d_1 + \cdots + d_N + \sum_{n=N+1}^{\infty} a_n \end{aligned}$$

(b) If $\sum a_n$ converged, part (a) would imply that $\sum d_n$ was also convergent.

63. Answers will vary.

64. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Differentiate:

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$$

Multiply by x:

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n$$

Differentiate:

$$\begin{aligned} \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) &= \frac{(1-x)^2(1) - (x)(2)(1-x)(-1)}{(1-x)^4} \\ &= \frac{(1-x)+2x}{(1-x)^3} = \frac{x+1}{(1-x)^3} \end{aligned}$$

$$\frac{x+1}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^{n-1}$$

Multiply by x:

$$\frac{x(x+1)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n$$

Let $x = \frac{1}{2}$:

$$\frac{\frac{1}{2}\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^3} = \sum_{n=0}^{\infty} n^2\left(\frac{1}{2}\right)^n$$

$$6 = \sum_{n=0}^{\infty}\frac{n^2}{2^n}$$

The sum is 6.

Section 9.5 Testing Convergence at Endpoints (pp. 513–525)

Exploration 1 The p-Series Test

1. We first note that the Integral Test applies to any series of the form $\sum\frac{1}{n^p}$ where p is positive. This is because the function $f(x) = x^{-p}$ is continuous and positive for all $x > 0$, and $f'(x) = -p \cdot x^{-p-1}$ is negative for all $x > 0$.

If $p > 1$:

$$\begin{aligned} \int_1^{\infty}\frac{1}{x^p}\,dx &= \lim_{k\to\infty}\int_1^k \frac{1}{x^p}\,dx = \lim_{k\to\infty}\left(\left[\frac{x^{-p+1}}{-p+1}\right]_1^k\right) \\ &= \lim_{k\to\infty}\left(\frac{1}{1-p} \cdot \left(\frac{1}{k^{p-1}} - 1\right)\right) \\ &= 0 + \frac{1}{p-1} \quad (\text{since } p-1 > 0) \\ &= \frac{1}{p-1} < \infty. \end{aligned}$$

The series converges by the Integral Test.