6. Yes, since each expression for an nth derivative given by the Quotient Rule will be a rational function whose denominator is a power of $x+1$.

7. No, since the function $f(x)=|x^2-4|$ has a corner at $x=2$.

8. Yes, since the derivatives of all orders for $\sin x$ and $\cos x$ are defined for all values of x.

9. Yes, since the function $f(x)=e^{-x}$ has derivatives of the form $f^{(n)}(x)=-e^{-x}$ for odd values of n and $f^{(n)}(x)=e^{-x}$ for even values of n, and both of these expressions are defined for all values of x.

10. No, since $f(x)=x^{3/2}$, we have $f'(x)=\frac{3}{2}x^{1/2}$ and $f''(x)=\frac{3}{4}x^{-1/2}$, so $f''(0)$ is undefined.

Section 9.3 Exercises

1. $$f(0)=e^{-2x}\Big|_{x=0}=1$$
$$f'(0)=-2e^{-2x}\Big|_{x=0}=-2$$
$$f''(0)=4e^{-2x}\Big|_{x=0}=4,\text{ so }\frac{f''(0)}{2!}=2$$
$$f'''(0)=-8e^{-2x}\Big|_{x=0}=-8,\text{ so }\frac{f'''(0)}{3!}=-\frac{4}{3}$$
$$f^{(4)}(0)=16e^{-2x}\Big|_{x=0}=16,\text{ so }\frac{f^{(4)}(0)}{4!}=\frac{2}{3}$$
$$P_4(x)=1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4$$
$$f(0.2)=P_4(0.2)=0.6704$$

2. $$f(0)=\cos\frac{\pi x}{2}\Big|_{x=0}=1$$
$$f'(0)=-\frac{\pi}{2}\sin\frac{\pi x}{2}\Big|_{x=0}=0$$
$$f''(0)=-\frac{\pi^2}{4}\cos\frac{\pi x}{2}\Big|_{x=0}=-\frac{\pi^2}{4},\text{ so }\frac{f''(0)}{2!}=-\frac{\pi^2}{8}$$
$$f'''(0)=\frac{\pi^3}{8}\sin\frac{\pi x}{2}\Big|_{x=0}=0,\text{ so }\frac{f'''(0)}{3!}=0$$
$$f^{(4)}(0)=\frac{\pi^4}{16}\cos\frac{\pi x}{2}\Big|_{x=0}=\frac{\pi^4}{16},\text{ so }\frac{f^{(4)}(0)}{4!}=\frac{\pi^4}{384}$$
$$P_4(x)=1-\frac{\pi^2}{8}x^2+\frac{\pi^4}{384}x^4$$
$$f(0.2)=P_4(0.2)\approx 0.9511$$

3. $$f(0)=5\sin(-x)\Big|_{x=0}=-5\sin x\Big|_{x=0}=0$$
$$f'(0)=-5\cos x\Big|_{x=0}=-5$$
$$f''(0)=5\sin x\Big|_{x=0}=0,\text{ so }\frac{f''(0)}{2!}=0$$
$$f'''(0)=5\cos x\Big|_{x=0}=5,\text{ so }\frac{f'''(0)}{3!}=\frac{5}{6}$$
$$f^{(4)}(0)=-5\sin x\Big|_{x=0}=0,\text{ so }\frac{f^{(4)}(0)}{4!}=0$$
$$P_4(x)=-5x+\frac{5}{6}x^3$$
$$f(0.2)\approx P_4(0.2)=-\frac{149}{150}\approx -0.9933$$

4. Substituting x^2 for x in the Maclaurin series given for $\ln(1+x)$ at the end of Section 9.2, we have
$$\ln(1+x^2)=x^2-\frac{(x^2)^2}{2}+\frac{(x^2)^3}{3}-\cdots+(-1)^{n-1}\frac{(x^2)^n}{n}+\cdots$$
$$=x^2-\frac{x^4}{2}+\frac{x^6}{3}-\cdots+(-1)^{n-1}\frac{x^{2n}}{n}+\cdots$$
Therefore, $P_4(x)=x^2-\frac{x^4}{2}$ and $f(0.2)\approx P_4(0.2)=0.0392$.

5. $$f(0)=(1-x)^{-2}\Big|_{x=0}=1$$
$$f'(0)=2(1-x)^{-3}\Big|_{x=0}=2$$
$$f''(0)=6(1-x)^{-4}\Big|_{x=0}=6,\text{ so }\frac{f''(0)}{2!}=3$$
$$f'''(0)=24(1-x)^{-5}\Big|_{x=0}=24,\text{ so }\frac{f'''(0)}{3!}=4$$
$$f^{(4)}(0)=120(1-x)^{-6}\Big|_{x=0}=120,\text{ so }\frac{f^{(4)}(0)}{4!}=5$$
$$P_4(x)=1+2x+3x^2+4x^3+5x^4$$
$$f(0.2)\approx P_4(0.2)=1.56$$

6. $$\sin x-x+\frac{x^3}{3!}$$
$$=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots\right)-x+\frac{x^3}{3!}$$
$$=\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots$$
Note: By replacing n with $n+2$, the general term can be written as $(-1)^n\frac{x^{2n+5}}{(2n+5)!}$

7. $xe^x = x\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots\right)$

$= x+x^2+\frac{x^3}{2!}+\cdots+\frac{x^{n+1}}{n!}+\cdots$

8. $\cos^2 x = \frac{1}{2}+\frac{1}{2}\cos\,(2x)$

$= \frac{1}{2}+\frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\cdots+(-1)^n\frac{(2x)^{2n}}{(2n)!}+\cdots\right)$

$= 1-\frac{4x^2}{2\cdot 2!}+\frac{16x^4}{2\cdot 4!}-\cdots+(-1)^n\frac{2^{2n}x^{2n}}{2\cdot(2n)!}+\cdots$

$= 1-x^2+\frac{x^4}{3}-\cdots+(-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!}+\cdots$

9. $\sin^2 x = \frac{1}{2}-\frac{1}{2}\cos(2x)$

$= \frac{1}{2}-\frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}\right.$

$\left.+\cdots+(-1)^n\frac{(2x)^{2n}}{(2n)!}+\cdots\right)$

$= \frac{4x^2}{2\cdot 2!}-\frac{16x^4}{2\cdot 4!}+\frac{64x^6}{2\cdot 6!}-\cdots+(-1)^{n-1}\frac{2^{2n}x^{2n}}{2\cdot(2n)!}+\cdots$

$= x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\cdots+(-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!}+\cdots$

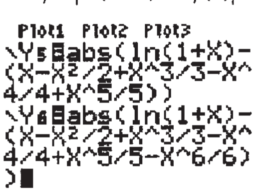
Note: By replacing n with n+1, the general term can be written as $(-1)^n\frac{2^{2n+1}x^{2n+2}}{(2n+2)!}$.

10. $\frac{x^2}{1-2x} = x^2\left(\frac{1}{1-2x}\right)$

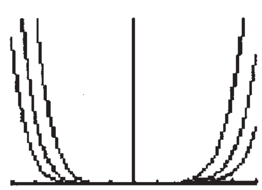
$= x^2[1+2x+(2x)^2+\cdots+(2x)^n+\cdots]$

$= x^2+2x^3+4x^4+\cdots+2^n x^{n+2}+\cdots$

11. $P_7(x)$. Use the window [–0.5, 0.5] by [0, 0.001] to graph $Y_n = |\ln(1+x)-P_n(x)|,\ n=1, 2, 3,\ldots$
The first graph that does not intersect the top of the window is $Y_7 = |\ln(1+x)-P_7(x)|$.

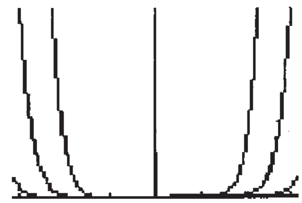


Formulas for Y5 and Y6



Graphs for Y5 and Y6 and Y7

12. $P_{12}(x)$. Use the window $[-\pi,\pi]$ by [0, 0.001] to graph $|\cos(x)-P_n(x)|,\ n=2, 4, 6,\ldots$
The first graph that does not intersect the top of the window is $|\cos(x)-P_{12}(x)|$.



Graphs of $|\cos(x)-P_8(x)|, |\cos(x)-P_{10}(x)|$ and $|\cos(x)-P_{12}(x)|$

13. $f(x)=\frac{1}{1-2x}=1+2x+(2x)^2+\cdots$

$|f(x)-P_6(x)| = |(2x)^7+(2x)^8+\cdots| = \left|\frac{(2x)^7}{1-2x}\right|$

For all $x\in\left(-\frac{1}{2},\frac{1}{2}\right)$, $1-2x>0$. Therefore the truncation error is: $\frac{2|x|^7}{1-2x}$

14. $\left|\frac{1}{1-x}-P_9(x)\right| = |x^{10}+x^{11}+\cdots| = \left|\frac{x^{10}}{1-x}\right|$

For all $x\in(-1, 1)$, $1-x>0$. Therefore the truncation error is: $\frac{x^{10}}{1-x}$

15. If $M=1$ and $r=1$, then $|\cos^{(n+1)}(t)| \le Mr^{n+1}=1$ for all t, since all derivatives of the cosine function are sine or cosine functions bounded between –1 and 1.

16. If $M=1$ and $r=5$, then $|f^{(n+1)}(t)| \le Mr^{n+1}=5^{n+1}$ for all t, since the (n + 1)st derivative of f is a sine or cosine function (bounded between –1 and 1) times 5^{n+1} (the result of $n+1$ applications of the Chain Rule).

17. If $M=1$ and $r=8$ then $|f^{(n+1)}(t)| \le Mr^{n+1}=8^{n+1}$ for all t, since the (n + 1)st derivative of f is a sine or cosine function (bounded between –1 and 1) times 8^{n+1} (the result of $n+1$ applications of the Chain Rule).

18. Note that $\left|f^{(n+1)}(t)\right| = \left|e^{5t} \cdot 5^{n+1}\right|$, which can be bounded for all t between 0 and x by $\left|e^{5x} \cdot 5^{n+1}\right|$ when $x > 0$ and by $\left|e^0 \cdot 5^{n+1}\right|$ when $x < 0$. Thus, let M be the larger of the two numbers $\{e^{5x}, 1\}$ and let $r = 5$.

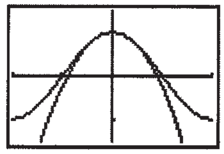
19. Let $f(x) = \sin x$. Then $P_4(x) = P_3(x) = x - \frac{x^3}{6}$, so we use the Remainder Estimation Theorem with $n = 4$. (We use P_4 instead of P_3 because it gives us a better range of x's.) Since $\left|f^{(5)}(x)\right| = |\cos x| \le 1$ for all x, we may use $M = r = 1$, giving $|R_4(x)| \le \frac{|x|^5}{5!}$, so we may assure that $|R_4(x)| \le 5 \times 10^{-4}$ by requiring $\frac{|x|^5}{5!} \le 5 \times 10^{-4}$, or $|x| \le \sqrt[5]{0.06} \approx 0.5697$. Thus, the absolute error is no greater than 5×10^{-4} when $-0.56 < x < 0.56$ (approximately).
Alternate method: Using graphing techniques,
$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \le 5 \times 10^{-4}$ when $-0.57 < x < 0.57$.

20. Let $f(x) = \cos x$. Then $P_3(x) = P_2(x) = 1 - \frac{x^2}{2}$, so we may use the Remainder Estimation Theorem with $n = 3$. (We use P_3 instead of P_2 because it gives us a better range of x's.) Since $\left|f^{(4)}(x)\right| = |\cos x| \le 1$ for all x, we may use $M = r = 1$, giving $|R_3(x)| \le \frac{|x|^4}{4!}$. For $|x| < 0.5$, the absolute error is less than $\frac{(0.5)^4}{4!} \approx 0.0026$ (approximately).

Alternate method: Using graphing techniques, we find that when $|x| < 0.5$,

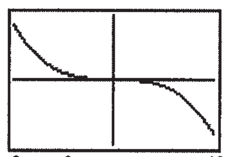
$$\begin{aligned}|\text{error}| &= \left|\cos x - \left(1 - \frac{x^2}{2}\right)\right| \\ &< \left|\cos 0.5 - \left(1 - \frac{0.5^2}{2}\right)\right| \\ &\approx 0.002583.\end{aligned}$$

The quantity $1 - \frac{x^2}{2}$ tends to be too small, as shown by the graphs of $y = \cos x$ and $y = 1 - \frac{x^2}{2}$.



$[-\pi, \pi]$ by $[-1.5, 1.5]$

21. Let $f(x) = \sin x$. Then $P_2(x) = P_1(x) = x$, so we may use the Remainder Estimation Theorem with $n = 2$. Since $|f'''(x)| = |-\cos x| \le 1$ for all x, we may use $M = r = 1$, giving $|R_2(x)| \le \frac{|x|^3}{3!}$. Thus, for $|x| < 10^{-3}$, the maximum possible error is about $\frac{(10^{-3})^3}{3!} \approx 1.67 \times 10^{-10}$.
Alternate method:
Using graphing techniques, we find that when $|x| < 10^{-3}$, $|\text{error}| = |\sin x - x| \le \left|\sin 10^{-3} - 10^{-3}\right| \approx 1.67 \times 10^{-10}$.
The inequality $x < \sin x$ is true for $x < 0$, as we may see by graphing $y = \sin x - x$.



$[-10^{-3}, 10^{-3}]$ by $[-2 \times 10^{-10}, 2 \times 10^{-10}]$

22. Let $f(x) = \sqrt{1+x}$. Then $P_1(x) = 1 + \frac{x}{2}$, so we may use the Remainder Estimation Theorem with $n = 1$. Since $|f''(x)| = \left|-\frac{1}{4}(1+x)^{-3/2}\right|$, which is less than 0.2538 for $|x| < 0.01$, we may use $M = 0.2538$ and $r = 1$, giving $|R_1(x)| \le \frac{0.2538|x|^2}{2!}$. Thus, for $|x| < 0.01$ the maximum possible absolute error is about $\frac{0.2538(0.01)^2}{2!} \approx 1.27 \times 10^{-5}$.

22. Continued

Alternate method:

Using graphing techniques, we find that when $|x| < 0.01$,

$$|\text{error}| = \left|\sqrt{1+x} - \left(1+\frac{x}{2}\right)\right|$$
$$\le \left|\sqrt{1-0.01} - \left(1-\frac{0.01}{2}\right)\right|$$
$$\approx 1.26\times 10^{-5}.$$

23. Note that $1+x+\frac{x^2}{2}$ is the second order Taylor polynomial for $f(x)=e^x$ at $x=0$, so we may use the Remainder Estimation Theorem with $n=2$. Since $|f'''(x)|=e^x$, which is less than $e^{0.1}$ when $|x|<0.1$, we may use $M=e^{0.1}$ and $r=1$, giving $|R_2(x)| \le \frac{e^{0.1}|x|^3}{3!}$. Thus, for $|x|<0.1$, the maximum possible error is about $\frac{e^{0.1}(0.1)^3}{3!} \approx 1.842\times 10^{-4}$.

24. Note that $e^x = 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$ and $e^{-x}=1-x+\frac{x^2}{2!}-\cdots+(-1)^n\frac{x^n}{n!}+\cdots$ Thus the terms with n even will cancel for $\sinh x = \frac{1}{2}(e^x-e^{-x})$, and the terms with n odd will cancel for $\cosh x = \frac{1}{2}(e^x-e^{-x})$.

$$\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots+\frac{x^{2n+1}}{(2n+1)!}+\ldots$$
$$\cosh x = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^{2n}}{(2n)!}+\ldots$$

25. All of the derivatives of $\cosh x$ are either $\cosh x$ or $\sinh x$. For any real x, $\cosh x$ and $\sinh x$ are both bounded by $e^{|x|}$. So for any real x, let $M=e^{|x|}$ and $r=1$ in the Remainder Estimation Theorem. This gives $|R_n(x)| \le \frac{e^{|x|}x^{n+1}}{(n+1)!}$ But for any fixed value of x, $\lim_{n\to\infty}\frac{e^{|x|}|x|^{n+1}}{(n+1)!}=0$. It follows that the series converges to $\cosh x$ for all real values of x.

26. For $n=0$, Taylor's Theorem with Remainder says that if f has derivatives of all orders in an open interval I containing a, then for each x in I, $f(x)=f(a)+R(x)$, where $R(x)=f'(c)(x-a)$, so $f(x)=f(a)+f'(c)(x-a)$ for some c between a and x. Letting $b=x$, this equation is $f(b)=f(a)+f'(c)(b-a)$, which is equivalent to $f'(c)=\frac{f(b)-f(a)}{b-a}$ for some c between a and b. Thus, for the class of functions that have derivatives of all orders in an open interval containing a and b, the Mean Value Theorem can be considered a special case of Taylor's Theorem.

27. $f(0)=\ln(\cos x)\big|_{x=0}=\ln 1=0$

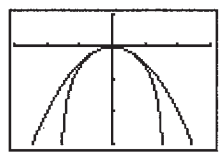
$f'(0)=\frac{1}{\cos x}(-\sin x)\big|_{x=0}=-\tan x\big|_{x=0}=0$

$f''(0)=-\sec^2 x\big|_{x=0}=-1$ so $\frac{f''(0)}{2!}=-\frac{1}{2}$

(a) $L(x)=0$

(b) $P_2(x)=-\frac{1}{2}x^2$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x=0$.



[–3, 3] by [–3, 1]

28. $f(0)=e^{\sin x}\big|_{x=0}=e^0=1$

$f'(0)=e^{\sin x}\cos x\big|_{x=0}=1$

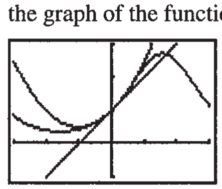
$f''(0)=\left[(e^{\sin x})(-\sin x)+(\cos x)(e^{\sin x}\cos x)\right]\Big|_{x=0}=1$,

so $\frac{f''(0)}{2!}=\frac{1}{2}$

(a) $L(x)=1+x$

(b) $P_2(x)=1+x+\frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x=0$.



[–3, 3] by [–1,3]

29. $f(0) = (1-x^2)^{-1/2}\big|_{x=0} = 1$

$f'(0) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)\big|_{x=0} = x(1-x^2)^{-3/2}\big|_{x=0} = 0$

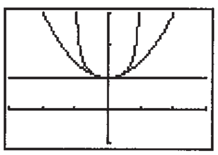
$f''(0) = (x)\left[-\frac{3}{2}(1-x^2)^{-5/2}(-2x)\right] + (1-x^2)^{-3/2}\big|_{x=0} = 1,$

so $\frac{f''(0)}{2!} = \frac{1}{2}$

(a) $L(x) = 1$

(b) $P_2(x) = 1 + \frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[−3, 3] by [−1, 3]

30. $f(0) = \sec x\big|_{x=0} = 1$

$f'(0) = \sec x \tan x\big|_{x=0} = 0$

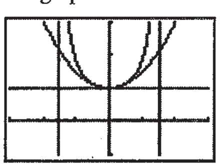
$f''(0) = (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x)\big|_{x=0} = 1,$

so $\frac{f''(0)}{2!} = \frac{1}{2}$

(a) $L(x) = 1$

(b) $P_2(x) = 1 + \frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[−3, 3] by [−1, 3]

31. $f(0) = \tan x\big|_{x=0} = 0$

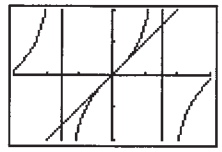
$f'(0) = \sec^2 x\big|_{x=0} = 1$

$f''(0) = (2\sec x)(\sec x \tan x)\Big|_{x=0} = 0$, so $\frac{f''(0)}{2!} = 0$

(a) $L(x) = x$

(b) $P_2(x) = x$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[−3, 3] by [−2, 2]

32. $f(0) = (1+x)^k\big|_{x=0} = 1$

$f'(0) = k(1+x)^{k-1}\big|_{x=0} = k$

$f''(0) = k(k-1)(1+x)^{k-2}\big|_{x=0} = k(k-1),$

so $\frac{f''(0)}{2!} = \frac{k(k-1)}{2}$

$P_2(x) = 1 + kx + \frac{k(k-1)}{2}x^2$

For $k = 3$, we have $f(x) = (1+x)^3$ and $f'''(x) = 6$. We may use the Remainder Estimation Theorem with $n = 2$, $M = 6$, and $r = 1$, giving $|R_2(x)| \le \frac{6|x|^3}{3!} = |x|^3$. (In this particular case it is actually true that $R_2(x) = x^3$, since $f(x)$ is a cubic polynomial.) Thus the absolute error is less than $\frac{1}{100}$ whenever $|x|^3 < 0.01$. In the interval [0, 1], this occurs when $0 \le x < \sqrt[3]{0.01} \approx 0.215$.

Alternate method:

Note that $P_2(x) = 1 + 3x + 3x^2$. Using graphing techniques, $\left|(1+x)^3 - (1+3x+3x^2)\right| < \frac{1}{100}$ when $|x| < 0.215$.

33. Let $f(x) = e^x$. Then $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, so we may use the Remainder Estimation Theorem with $n = 3$. Since $\left|f^{(4)}(x)\right| = e^x$, which is no more than $e^{0.1}$ when $|x| \le 0.1$, we may use $M = e^{0.1}$ and $r = 1$, giving $|R_3(x)| \le \frac{e^{0.1}|x|^4}{4!}$.

Thus, for $|x| \le 0.1$, the maximum possible absolute error is about $\frac{e^{0.1}(0.1)^4}{24} \approx 4.605 \times 10^{-6}$.

Alternate method:

Using graphing techniques, when $|x| \le 0.1$,

$$|\text{error}| = \left|e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\right|$$

$$\le \left|e^{0.1} - \left(1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6}\right)\right|$$

$$\approx 4.251 \times 10^{-6}.$$

34. Since the Maclaurin series is

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots,$$

$$P_3(x)=1+x+x^2+x^3.$$

Since $\left|f^{(4)}(x)\right|=24(1-x)^{-5}$, which is no more than $24(0.9)^{-5}$ when $|x|\le 0.1$, we may use $M=24(0.9)^{-5}$ and $r=1$, giving $|R_3(x)|\le\frac{24(0.9)^{-5}|x|^4}{4!}=\frac{|x|^4}{0.9^5}$. Thus, for $|x|\le 0.1$, an upper bound for the magnitude of the approximation error is $\frac{0.1^4}{0.9^5}\approx 1.694\times 10^{-4}$. Rounding up to be safe, an upper bound is 1.70×10^{-4}.

Alternate method:

Using graphing techniques, when $|x|\le 0.1$,

$$|\text{error}|=\left|\frac{1}{1-x}-(1+x+x^2+x^3)\right|$$

$$\le\left|\frac{1}{1-0.1}-1.111\right|\approx 1.11\times 10^{-4}.$$

35. (a) No

(b) Yes, since

$$\frac{dy}{dx}=e^{-x^2}=1+(-x^2)+\frac{(-x^2)^2}{2!}+\cdots+\frac{(-x^2)^n}{n!}+\cdots$$

$$=1-x^2+\frac{x^4}{2!}-\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots.$$

The constant term of y is $y(0)=2$, and we may obtain the remaining terms of y by integrating the above series.

$$y=2+x-\frac{x^3}{3}+\frac{x^5}{10}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+\cdots$$

By substituting $n-1$ for n, the general term may also be written as $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)(n-1)!}$.

(c) The power series equals the function y for all real values of x. This is because the series for e^{-x^2} converges for all real values of x, so Theorem 2 of Section 9.1 implies that the new series also converges for all x.

36. (a) Substitute $-x$ for x in the Maclaurin series for $\ln(1+x)$ given at the end of Section 9.2.

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots-\frac{x^n}{n}-\cdots$$

(b)

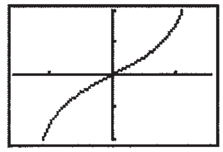
$$\ln\frac{1+x}{1-x}=\ln(1+x)-\ln(1-x)$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots\right)$$

$$+\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}\right)$$

$$=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\cdots+\frac{2x^{2n+1}}{2n+1}+\cdots$$

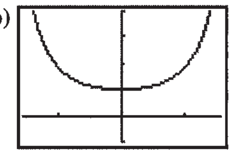
37. (a)



$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ by [–2, 2]

The series approximates $\tan x$.

(b)



$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ by [–1, 4]

The series approximates $\sec x$.

38. False. If $f'(a)$ happens to be 0, the linearization is a constant function.

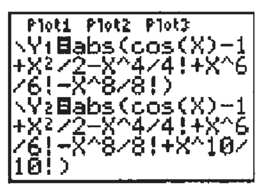
39. True. The coefficient of x is $f'(0)$.

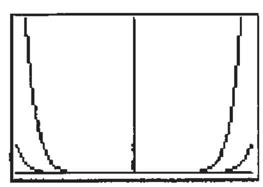
40. D. $1.5-\frac{1.5^3}{3!}+\frac{1.5^5}{5!}\approx 1.001$

41. E. Coefficient of $x^{12}=\frac{f^{(12)}(0)}{12!}=\frac{1}{11!}$,

$$f^{(12)}(0)=\frac{12!}{11!}=12$$

42. B. Consider the graphs of $Y_1=|P_8(x)-\cos(x)|$ and $Y_2=|P_{10}(x)-\cos(x)|$ in the window $[-\pi,\pi]$ by $[0, 0.01]$. The graph of y_1 intersects the top of the window, but the graph of y_{10} does not, so 10 is the smallest value of n for which the maximum possible error on the interval $[-\pi,\pi]$ is less than 0.01.





43. A.

44. (a) $\sin^2(x)=\frac{1}{2}(1-\cos 2x)$

$$=\frac{1}{2}-\frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\cdots+(-1)^n\frac{(2x)^{2n}}{(2n)!}+\cdots\right)$$

$$=\frac{4x^2}{2\cdot 2!}-\frac{16x^4}{2\cdot 4!}+\frac{64x^6}{2\cdot 6!}-\frac{256x^8}{2\cdot 8!}+\cdots-(-1)^n\frac{2^{2n}x^{2n}}{2\cdot(2n)!}+\cdots$$

$$=x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\frac{x^8}{315}+\cdots+(-1)^{n+1}\frac{2^{2n-1}x^{2n}}{(2n)!}+\cdots$$

(b) $2\sin x\cos x=\frac{d}{dx}\left(x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\frac{x^8}{315}+\cdots+(-1)^{n+1}\frac{2^{2n-1}x^{2n}}{(2n)!}+\cdots\right)$

$$=2x-\frac{4x^3}{3}+\frac{12x^5}{45}-\frac{8x^7}{315}+\cdots+(-1)^{n+1}\frac{2^{2n-1}\cdot 2n\cdot x^{2n-1}}{(2n)!}+\cdots$$

$$=2x-\frac{4x^3}{3}+\frac{4x^5}{15}-\frac{8x^7}{315}+\cdots+(-1)^{n+1}\frac{2^{2n-1}\cdot x^{2n-1}}{(2n-1)!}+\cdots$$

(c) Writing out the first few terms of the series for sin $(2x)$ and simplifying will show that the beginning terms match. The tricky part is matching up the general terms. It is helpful to take the series we got in part (b) and carry it out to the $(n+1)^{\text{st}}$ term:

$$2\sin x\cos x=2x-\frac{4x^3}{3}+\cdots+(-1)^{n+1}\frac{2^{2n-1}\cdot x^{2n-1}}{(2n-1)!}+(-1)^{(n+1)+1}\frac{2^{2(n+1)-1}\cdot x^{2(n+1)-1}}{(2(n+1)-1)!}+\cdots$$

$$=2x-\frac{4x^3}{3}+\cdots+(-1)^{n+1}\frac{2^{2n-1}\cdot x^{2n-1}}{(2n-1)!}+(-1)^{n+2}\frac{2^{2n+1}\cdot x^{2n-1}}{(2n+1)!}+\cdots$$

$$=2x-\frac{4x^3}{3}+\cdots+(-1)^{n-1}\frac{(2x)^{2n-1}}{(2n-1)!}+(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}+\cdots$$

$$=\sin 2x$$

45. (a) It works. For example, let $n=2$. Then $P=3.14$ and $P+\sin P\approx 3.141592653$, which is accurate to more than 6 decimal places.

(b) Let $P=\pi+x$ where x is the error in the original estimate. Then
$P+\sin P=(\pi+x)+\sin(\pi+x)=\pi+x-\sin x$

But by the Remainder Theorem, $|x-\sin x|<\frac{|x|^3}{6}$.

Therefore, the difference between the new estimate $P+\sin P$ and π is less than $\frac{|x|^3}{6}$.

46. (a) $$\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{(\cos\theta+i\sin\theta)+(\cos(-\theta)+i\sin(-\theta))}{2}=\frac{\cos\theta+i\sin\theta+\cos\theta-i\sin\theta}{2}=\frac{2\cos\theta}{2}=\cos\theta$$

(b) $$\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{(\cos\theta+i\sin\theta)-(\cos(-\theta)+i\sin(-\theta))}{2i}=\frac{(\cos\theta+i\sin\theta)-(\cos\theta-i\sin\theta)}{2i}=\frac{2i\sin\theta}{2i}=\sin\theta$$

47. $$\frac{d}{dx}\left[e^{ax}(\cos bx+i\sin bx)\right]$$
$$=(e^{ax})(-b\sin bx+ib\cos bx)+(ae^{ax})(\cos bx+i\sin bx)$$
$$=(e^{ax})\left[(i^2b\sin bx+ib\cos bx)+a(\cos bx+i\sin bx)\right]$$
$$=(e^{ax})\left[ib(\cos bx+i\sin bx)+a(\cos bx+i\sin bx)\right]$$
$$=(a+ib)(e^{ax})(\cos bx+i\sin bx)$$
$$=(a+ib)e^{(a+ib)x}$$

48. (a) The derivative of the right-hand side is

$$\frac{a-ib}{a^2+b^2}(a+ib)e^{(a+ib)x}=\frac{a^2-(ib)^2}{a^2+b^2}e^{(a+ib)x}=\frac{a^2+b^2}{a^2+b^2}e^{(a+ib)x}=e^{(a+ib)x},$$

which confirms the antiderivative formula.

48. Continued

(b) $\int e^{ax}\cos bx\,dx + i\int e^{ax}\sin bx\,dx$

$$= \int e^{(a+ib)x}\,dx$$

$$= \frac{a-ib}{a^2+b^2}e^{(a+ib)x}$$

$$= \frac{a-ib}{a^2+b^2}e^{ax}(\cos bx + i\sin bx)$$

$$= \left(\frac{e^{ax}}{a^2+b^2}\right)(a\cos bx + b\sin bx - ib\cos bx + ia\sin bx)$$

$$= \left(\frac{e^{ax}}{a^2+b^2}\right)[(a\cos bx + b\sin bx) + i(a\sin bx - b\cos bx)]$$

Separating the real and imaginary parts gives

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) \text{ and}$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Quick Quiz Sections 9.1–9.3

1. D. $\displaystyle\sum_{n=0}^{\infty}\frac{\pi^n}{e^{2n}} = \sum_{n=0}^{\infty}\left(\frac{\pi}{e^2}\right)^n = \frac{1}{1+\frac{\pi}{e^2}} = \frac{e^2}{e^2+\pi}$

2. A. $2 - 1x + \dfrac{6x^2}{2!} + \dfrac{12x^3}{3!} = 2 - x + 3x^2 + 2x^3$

3. E. $\dfrac{1}{x} = \dfrac{1}{1+(x-1)}$

Substitute $(x-1)$ into the Maclaurin series for $\dfrac{1}{1+x}$ to obtain the series: $\displaystyle\sum_{n=0}^{\infty}(-1)^n(x-1)^n$

4. (a) Since the series is geometric, it converges if and only if $|r| < 1$, where $r = \dfrac{x+2}{3}$.

$\left|\dfrac{x+2}{3}\right| < 1 \Rightarrow |x+2| < 3 \Rightarrow -5 < x < 1$. The interval of convergence is $(-5, 1)$.

(b) The series is geometric with first term 2 and common ratio $r = \dfrac{x+2}{3}$. It therefore converges to $\dfrac{2}{1-\frac{x+2}{3}} = \dfrac{6}{1-x}$.

Section 9.4 Radius of Convergence (pp. 503–512)

Exploration 1 Finishing the Proof of the Ratio Test

1. For $\sum \frac{1}{n}$: $L = \lim\limits_{n\to\infty}\dfrac{\frac{1}{n+1}}{\frac{1}{n}} = \lim\limits_{n\to\infty}\dfrac{n}{n+1} = 1$.

For $\sum \frac{1}{n^2}$: $L = \lim\limits_{n\to\infty}\dfrac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty}\dfrac{n^2}{(n+1)^2} = 1$.

2. (a) $\displaystyle\int_1^{\infty}\frac{1}{x}\,dx = \lim_{k\to\infty}\left(\ln x\Big|_1^k\right) = \lim_{k\to\infty}\ln k = \infty.$

(b) $\displaystyle\int_1^{\infty}\frac{1}{x^2}\,dx = \lim_{k\to\infty}\left(-x^{-1}\Big|_1^k\right) = \lim_{k\to\infty}\left(-\frac{1}{k}+1\right) = 1.$

3. Figure 9.14a shows that $\sum\frac{1}{n}$ is greater than $\int_1^{\infty}\frac{1}{x}\,dx$. Since the integral diverges, so must the series.

Figure 9.14b shows that $\sum\frac{1}{n^2}$ is less than $1 + \int_1^{\infty}\frac{1}{x^2}\,dx$. Since the integral converges, so must the series.

4. These two examples prove that $L = 1$ can be true for either a divergent series or a convergent series. The Ratio Test itself is therefore inconclusive when $L = 1$.

Exploration 2 Revisiting a Maclaurin Series

1. $L = \lim\limits_{n\to\infty}\dfrac{|x|^{n+1}}{n+1}\cdot\dfrac{n}{|x|^n} = \lim\limits_{n\to\infty}\dfrac{n}{n+1}|x| = |x|$. The series converges absolutely when $|x| < 1$, so the radius of convergence is 1.

2. When $x = -1$, the series becomes

$$-1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} - \cdots.$$

Each term in this series is the negative of the corresponding term in the divergent series of Figure 9.14a. Just as $\sum\frac{1}{n}$ diverges to $+\infty$, this series diverges to $-\infty$.