5. $f(x)=x^n$

$$f'(x)=nx^{n-1}$$
$$f''(x)=n(n-1)x^{n-2}$$
$$f'''(x)=n(n-1)(n-2)x^{n-3}$$
$$f^{(k)}(x)=\frac{n!}{(n-k)!}x^{n-k}$$
$$f^{(n)}(x)=\frac{n!}{0!}x^0=n!$$

6. $\frac{dy}{dx}=\frac{d}{dx}\frac{x^n}{n!}=\frac{nx^{n-1}}{n!}=\frac{x^{n-1}}{(n-1)!}$

7. $\frac{dy}{dx}=\frac{d}{dx}\frac{2^n(x-a)^n}{n!}=\frac{2^n n(x-a)^{n-1}}{n!}=\frac{2^n(x-a)^{n-1}}{(n-1)!}$

8. $\frac{dy}{dx}=\frac{d}{dx}\left[(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right]=(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}=\frac{(-1)^n x^{2n}}{(2n)!}$

9. $\frac{dy}{dx}=\frac{d}{dx}\frac{(x+a)^{2n}}{(2n)!}=\frac{2n(x+a)^{2n-1}}{(2n)!}=\frac{(x+a)^{2n-1}}{(2n-1)!}$

10. $\frac{dy}{dx}=\frac{d}{dx}\frac{(1-x)^n}{n!}=\frac{n(1-x)^{n-1}(-1)}{n!}$

$$=-\frac{(1-x)^{n-1}}{(n-1)!}$$

Section 9.2 Exercises

1. $f(0)=\sqrt{1+x^2}\Big|_{x=0}=1$

$$f'(0)=\frac{x}{\sqrt{1+x^2}}\Big|_{x=0}=0$$
$$f''(0)=\frac{1}{(1+x^2)^{3/2}}\Big|_{x=0}=1$$
$$f'''(0)=\frac{-3x}{(1+x^2)^{5/2}}\Big|_{x=0}=0$$
$$f^{(4)}(0)=\frac{12x^2-3}{(1+x^2)^{7/2}}\Big|_{x=0}=-3$$
$$P_4(x)=1+0x+\frac{1}{2}x^2+\frac{0}{6}x^3+\frac{-3}{24}x^4$$
$$=1+\frac{1}{2}x^2-\frac{1}{8}x^4$$

2. $f(0)=e^{2x}\Big|_{x=0}=1$

$$f'(0)=2e^{2x}\Big|_{x=0}=2$$
$$f''(0)=4e^{2x}\Big|_{x=0}=4$$
$$f'''(0)=8e^{2x}\Big|_{x=0}=8$$
$$f^{(4)}(0)=16e^{2x}\Big|_{x=0}=16$$
$$P_4(x)=1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4$$

3. $f(0)=\frac{1}{x+2}\Big|_{x=0}=\frac{1}{2}$

$$f'(0)=\frac{-1}{(x+2)^2}\Big|_{x=0}=-\frac{1}{4}$$
$$f''(0)=\frac{2}{(x+2)^3}\Big|_{x=0}=\frac{1}{4}$$
$$f'''(0)=-\frac{-6}{(x+2)^4}\Big|_{x=0}=-\frac{3}{8}$$
$$f^{(4)}(0)=\frac{24}{(x+2)^5}\Big|_{x=0}=\frac{3}{4}$$
$$f^{(5)}(0)=\frac{-120}{(x+2)^6}\Big|_{x=0}=-\frac{15}{8}$$
$$P_5(x)=\frac{1}{2}-\frac{1}{4}x+\frac{1}{4}\bullet\frac{1}{2}x^2-\frac{3}{8}\bullet\frac{1}{6}x^3$$
$$+\frac{3}{4}\bullet\frac{1}{24}x^4-\frac{15}{8}\bullet\frac{1}{120}x^5$$
$$=\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2-\frac{1}{16}x^3+\frac{1}{32}x^4-\frac{1}{64}x^5$$
$$\frac{1}{x+2}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1}x^n$$

4. $f(0)=e^{1-x}\Big|_{x=0}=e$

$$f'(0)=-e^{1-x}\Big|_{x=0}=-e$$
$$f''(0)=e^{1-x}\Big|_{x=0}=e$$
$$f'''(0)=-e^{1-x}\Big|_{x=0}=-e$$
$$f^{(4)}(0)=e^{1-x}\Big|_{x=0}=e$$
$$f^{(5)}(0)=-e^{1-x}\Big|_{x=0}=-e$$
$$P_5(x)=e-ex+\frac{e}{2}x^2-\frac{e}{6}x^3+\frac{e}{24}x^4-\frac{e}{120}x^5$$
$$e^{1-x}=\sum_{n=0}^{\infty}(-1)^n\frac{e}{n!}x^n$$

5. Substitute $2x$ for x in the Maclaurin series for $\sin x$ shown at the end of Section 9.2.

$$\sin 2x=2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\cdots+(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}+\cdots$$
$$=2x-\frac{4x^3}{3}+\frac{4x^5}{15}-\cdots+\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}+\cdots$$

This series converges for all real x.

6. Substitute $-x$ for x in the Maclaurin series for $\ln(1+x)$ shown at the end of Section 9.2.

$$\ln(1-x)=(-x)-\frac{(-x)^2}{2}+\frac{(-x)^3}{3}-\cdots+(-1)^{n-1}\frac{(-x)^n}{n}+\cdots$$
$$=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots-\frac{x^n}{n}-\cdots$$

This series converges when $-1<-x\le 1$, so the interval of convergence is $[-1, 1)$.

7. Substitute x^2 for x in the Maclaurin series for $\tan^{-1} x$ shown at the end of Section 9.2.

$$\tan^{-1} x^2 = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \cdots + (-1)^n \frac{(x^2)^{2n+1}}{2n+1} + \cdots$$
$$= x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \cdots + \frac{(-1)^n x^{4n+2}}{2n+1} + \cdots$$

This series converges when $|x^2| \le 1$, so the interval of convergence is $[-1, 1]$.

8. $$7xe^x = 7x\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$
$$= 7x + 7x^2 + \frac{7x^3}{2!} + \cdots + \frac{7x^{n+1}}{n!}$$

This series converges for all real x.

9. $$\cos(x+2) = (\cos 2)(\cos x) - (\sin 2)(\sin x)$$
$$= (\cos 2)\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}\right]$$
$$- (\sin 2)\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right]$$
$$= (\cos 2) - (\sin 2)x - \frac{(\cos 2)x^2}{2!} + \frac{(\sin 2)x^3}{3!}$$
$$+ \frac{(\cos 2)x^4}{4!} - \frac{(\sin 2)x^5}{5!} + \cdots + (-1)^n \frac{(\cos 2)x^{2n}}{(2n)!}$$
$$+ (-1)^{n+1} \frac{(\sin 2)x^{2n+1}}{(2n+1)!} + \cdots$$

We need to find an expression for the coefficient of x^k. Note that the coefficient depends on whether k is even or odd. We can use the greatest integer function to simplify the formula:

$$\text{Coefficient of } x^k = \begin{cases} \dfrac{(-1)^n (\cos 2)}{(2n)!} & k = 2n \\ \dfrac{(-1)^{n+1} (\sin 2)}{(2n+1)!} & k = 2n+1 \end{cases}$$
$$= \begin{cases} \dfrac{(-1)^{\text{int}[(k+1)/2]} (\cos 2)}{k!} & k \text{ even} \\ \dfrac{(-1)^{\text{int}[(k+1)/2]} (\sin 2)}{k!} & k \text{ odd} \end{cases}$$

Turn 'k' back into 'n' to get the general term:

$$\cos(x+2) = (\cos 2) - (\sin 2)x - \frac{(\cos 2)x^2}{2!}$$
$$+ \cdots + \frac{(-1)^{A_n} B_n x^n}{n!} + \cdots$$

where $A_n = \text{int}\left[\dfrac{n+1}{2}\right]$ and $B_n = \begin{cases} \cos 2 & n \text{ even} \\ \sin 2 & n \text{ odd} \end{cases}$

We can get an alternate form of the series if we observe that:

$$-\sin 2 = \cos\left(2 + \frac{\pi}{2}\right), \quad -\cos 2 = \cos(2 + \pi),$$
$$\sin 2 = \cos\left(2 + \frac{3\pi}{2}\right) \quad \text{and so on.}$$

The series becomes:

$$\cos(x+2) = (\cos 2) + \cos\left(2 + \frac{\pi}{2}\right)x + \cos(2+\pi)\frac{x^2}{2!}$$
$$+ \cdots + \cos\left(2 + \frac{n\pi}{2}\right)\frac{x^n}{n!} + \cdots$$

10. $$x^2 \cos x = x^2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right)$$
$$= x^2 - \frac{x^4}{2} + \frac{x^6}{24} - \cdots + \frac{(-1)^n x^{2n+2}}{(2n)!} + \cdots$$

The series converges for all real x.

11. Factor out x and substitute x^3 for x in the Maclaurin series for $\dfrac{1}{1-x}$ shown at the end of Section 9.2.

$$\frac{x}{1-x^3} = x\left(\frac{1}{1-x^3}\right)$$
$$= x[1 + x^3 + (x^3)^2 + \cdots + (x^3)^n + \cdots]$$
$$= x + x^4 + x^7 + \cdots + x^{3n+1} + \cdots$$

The series converges for $|x^3| < 1$, so the interval of convergence is $(-1, 1)$.

12. Substitute $-2x$ for x in the Maclaurin series for e^x shown at the end of Section 9.2.

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \cdots + \frac{(-2x)^n}{n!} + \cdots$$
$$= 1 - 2x + 2x^2 - \cdots + \frac{(-1)^n 2^n x^n}{n!} + \cdots$$

The series converges for all real x.

13. $$f(2) = \left.\frac{1}{x+1}\right|_{x=2} = \frac{1}{3}$$
$$f'(2) = \left.\frac{-1}{(x+1)^2}\right|_{x=2} = -\frac{1}{9}$$
$$f''(2) = \left.\frac{2}{(x+1)^3}\right|_{x=2} = \frac{2}{27}$$
$$\vdots$$
$$f^{(n)}(2) = \left.\frac{(-1)^n n!}{(x+1)^{n+1}}\right|_{x=2} = \frac{(-1)^n n!}{3^{n+1}}$$

$$\frac{1}{x+1} = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{2}{27}\frac{(x-2)^2}{2!}$$
$$+ \cdots + \frac{(-1)^n n!}{3^{n+1}}\frac{(x-2)^n}{n!} + \cdots$$
$$= \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2$$
$$+ \cdots + \frac{(-1)^n}{3^{n+1}}(x-2)^n + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x-2)^n$$

14. $f(1)=e^{x/2}\Big|_{x=1}=e^{1/2}$

$f'(1)=\frac{1}{2}e^{x/2}\Big|_{x=1}=\frac{1}{2}e^{1/2}$

$f''(1)=\frac{1}{4}e^{x/2}\Big|_{x=1}=\frac{1}{4}e^{1/2}$

$\vdots$

$f^{(n)}(1)=\left(\frac{1}{2}\right)^n e^{x/2}\Big|_{x=1}=\left(\frac{1}{2}\right)^n e^{1/2}$

$$e^{x/2}=e^{1/2}+\frac{1}{2}e^{1/2}(x-1)+\frac{1}{4}e^{1/2}\bullet\frac{(x-1)^2}{2}+\cdots+\left(\frac{1}{2}\right)^n e^{1/2}\bullet\frac{(x-1)^n}{n!}+\cdots$$

$$=e^{1/2}+e^{1/2}\frac{(x-1)}{2}+e^{1/2}\frac{(x-1)^2}{8}+\cdots+e^{1/2}\frac{(x-1)^n}{2^n n!}+\cdots$$

$$=\sum_{n=0}^{\infty}e^{1/2}\frac{(x-1)^n}{2^n n!}$$

15. (a) Since f is a cubic polynomial, it is its own Taylor polynomial of order 3.

$P_3(x)=x^3-2x+4=4-2x+x^3$

(b) $f(1)=x^3-2x+4\Big|_{x=1}=3$

$f'(1)=3x^2-2\Big|_{x=1}=1$

$f''(1)=6x\Big|_{x=1}=6$, so $\frac{f''(1)}{2!}=3$

$f'''(1)=6\Big|_{x=1}=6$, so $\frac{f'''(1)}{3!}=1$

$P_3(x)=3+(x-1)+3(x-1)^2+(x-1)^3$

16. (a) Since f is a cubic polynomial, it is its own Taylor polynomial of order 3.

$P_3(x)=2x^3+x^2+3x-8=-8+3x+x^2+2x^3$

(b) $f(1)=2x^3+x^2+3x-8\Big|_{x=1}=-2$

$f'(1)=6x^2+2x+3\Big|_{x=1}=11$

$f''(1)=12x+2\Big|_{x=1}=14$, so $\frac{f''(1)}{2!}=7$

$f'''(1)=12\Big|_{x=1}=12$, so $\frac{f'''(1)}{3!}=2$

$P_3(x)=-2+11(x-1)+7(x-1)^2+2(x-1)^3$

17. (a) Since $f(0)=f'(0)=f''(0)=f'''(0)=0$, the Taylor polynomial of order 3 is $P_3(0)=0$.

(b) $f(1)=x^4\Big|_{x=1}=1$

$f'(1)=4x^3\Big|_{x=1}=4$

$f''(1)=12x^2\Big|_{x=1}=12$, so $\frac{f''(1)}{2!}=6$

$f'''(1)=24x\Big|_{x=1}=24$, so $\frac{f'''(1)}{3!}=4$

$P_3(x)=1+4(x-1)+6(x-1)^2+4(x-1)^3$

18. $f(2)=\frac{1}{x}\Big|_{x=2}=\frac{1}{2}$

$f'(2)=-x^{-2}\Big|_{x=2}=-\frac{1}{4}$

$f''(2)=2x^{-3}\Big|_{x=2}=\frac{1}{4}$, so $\frac{f''(2)}{2!}=\frac{1}{8}$

$f'''(2)=-6x^{-4}\Big|_{x=2}=-\frac{3}{8}$, so $\frac{f'''(2)}{3!}=-\frac{1}{16}$

$P_0(x)=\frac{1}{2}$

$P_1(x)=\frac{1}{2}-\frac{x-2}{4}$

$P_2(x)=\frac{1}{2}-\frac{x-2}{4}+\frac{(x-2)^2}{8}$

$P_3(x)=\frac{1}{2}-\frac{x-2}{4}+\frac{(x-2)^2}{8}-\frac{(x-2)^3}{16}$

19. $f\left(\frac{\pi}{4}\right)=\sin x\Big|_{x=\pi/4}=\frac{\sqrt{2}}{2}$

$f'\left(\frac{\pi}{4}\right)=\cos x\Big|_{x=\pi/4}=\frac{\sqrt{2}}{2}$

$f''\left(\frac{\pi}{4}\right)=-\sin x\Big|_{x=\pi/4}=-\frac{\sqrt{2}}{2}$, so $\frac{f''\left(\frac{\pi}{4}\right)}{2!}=-\frac{\sqrt{2}}{4}$

$f'''\left(\frac{\pi}{4}\right)=-\cos x\Big|_{x=\pi/4}=-\frac{\sqrt{2}}{2}$, so $\frac{f'''\left(\frac{\pi}{4}\right)}{3!}=-\frac{\sqrt{2}}{12}$

$P_0(x)=\frac{\sqrt{2}}{2}$

$P_1(x)=\frac{\sqrt{2}}{2}+\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)$

$P_2(x)=\frac{\sqrt{2}}{2}+\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)-\left(\frac{\sqrt{2}}{4}\right)\left(x-\frac{\pi}{4}\right)^2$

$$P_3(x)=\frac{\sqrt{2}}{2}+\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)-\left(\frac{\sqrt{2}}{4}\right)\left(x-\frac{\pi}{4}\right)^2-\left(\frac{\sqrt{2}}{12}\right)\left(x-\frac{\pi}{4}\right)^3$$

20. $f\left(\frac{\pi}{4}\right)=\cos x\Big|_{x=\pi/4}=\frac{\sqrt{2}}{2}$

$f'\left(\frac{\pi}{4}\right)=-\sin x\Big|_{x=\pi/4}=-\frac{\sqrt{2}}{2}$

$f''\left(\frac{\pi}{4}\right)=-\cos x\Big|_{x=\pi/4}=-\frac{\sqrt{2}}{2}$, so $\frac{f''\left(\frac{\pi}{4}\right)}{2!}=-\frac{\sqrt{2}}{4}$

$f'''\left(\frac{\pi}{4}\right)=\sin x\Big|_{x=\pi/4}=\frac{\sqrt{2}}{2}$, so $\frac{f'''\left(\frac{\pi}{4}\right)}{3!}=\frac{\sqrt{2}}{12}$

$P_0(x)=\frac{\sqrt{2}}{2}$

$P_1(x)=\frac{\sqrt{2}}{2}-\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)$

$P_2(x)=\frac{\sqrt{2}}{2}-\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)-\left(\frac{\sqrt{2}}{4}\right)\left(x-\frac{\pi}{4}\right)^2$

$P_3(x)=\frac{\sqrt{2}}{2}-\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)-\left(\frac{\sqrt{2}}{4}\right)\left(x-\frac{\pi}{4}\right)^2+\left(\frac{\sqrt{2}}{12}\right)\left(x-\frac{\pi}{4}\right)^3$

21. $f(4)=x^{1/2}\Big|_{x=4}=2$

$f'(4)=\frac{1}{2}x^{-1/2}\Big|_{x=4}=\frac{1}{4}$

$f''(4)=-\frac{1}{4}x^{-3/2}\Big|_{x=4}=-\frac{1}{32}$, so $\frac{f''(4)}{2!}=-\frac{1}{64}$

$f'''(4)=\frac{3}{8}x^{-5/2}\Big|_{x=4}=\frac{3}{256}$, so $\frac{f'''(4)}{3!}=\frac{1}{512}$

$P_0(x)=2$

$P_1(x)=2+\frac{x-4}{4}$

$P_2(x)=2+\frac{x-4}{4}-\frac{(x-4)^2}{64}$

$P_3(x)=2+\frac{x-4}{4}-\frac{(x-4)^2}{64}+\frac{(x-4)^3}{512}$

22. (a) $P_3(x)=4+5x+\frac{-8}{2!}x^2+\frac{6}{3!}x^3$

$=4+5x-4x^2+x^3$

$f(0.2)\approx P_3(0.2)=4.848$

(b) Since the Taylor series of $f'(x)$ can be obtained by differentiating the terms of the Taylor series of $f(x)$, the second order Taylor polynomial of $f'(x)$ is given by $5-8x+3x^2$. Evaluating at $x=0.2$, $f'(0.2)\approx 3.52$

23. (a) $P_3(x)=4+(-1)(x-1)+\frac{3}{2!}(x-1)^2+\frac{2}{3!}(x-1)^3$

$=4-(x-1)+\frac{3}{2}(x-1)^2+\frac{1}{3}(x-1)^3$

$f(1.2)\approx P_3(1.2)\approx 3.863$

(b) Since the Taylor series of $f'(x)$ can be obtained by differentiating the terms of the Taylor series of $f(x)$, the second order Taylor polynomial of $f'(x)$ is given by $-1+3(x-1)+(x-1)^2$. Evaluating at $x=1.2$, $f'(1.2)\approx -0.36$

24. (a) Since $f'(0)x=\frac{x}{2!}$, $f'(0)=\frac{1}{2!}=\frac{1}{2}$.

Since $\frac{f^{(10)}(0)}{10!}x^{10}=\frac{x^{10}}{11!}$, $f^{(10)}(0)=\frac{10!}{11!}=\frac{1}{11}$.

(b) Multiply each term of $f(x)$ by x.

$g(x)=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^{n+1}}{(n+1)!}+\cdots$

(c) $g(x)=e^x-1$

25. (a) Substitute $\frac{x}{2}$ for x in the Maclaurin series for e^x shown at the end of Section 9.2

$e^{x/2}=1+\frac{x}{2}+\frac{\left(\frac{x}{2}\right)^2}{2}+\cdots+\frac{\left(\frac{x}{2}\right)^n}{n!}+\cdots$

$=1+\frac{x}{2}+\frac{x^2}{8}+\cdots+\frac{x^n}{2^n\cdot n!}$

(b) $g(x)=\frac{e^x-1}{x}$

$=\frac{1}{x}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots\right)-1\right]$

$=\frac{1}{x}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots\right)$

$=1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots+\frac{x^{n-1}}{n!}+\cdots$

This can also be written as

$1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots+\frac{x^n}{(n+1)!}+\cdots.$

25. Continued

(c) $g'(x) = \frac{d}{dx}\left(\frac{e^x - 1}{x}\right) = \frac{(x)(e^x) - (e^x - 1)(1)}{x^2}$

$= \frac{xe^x - e^x + 1}{x^2}$

$g'(1) = \frac{e - e + 1}{1} = 1$

From the series,

$g'(x) = \frac{d}{dx}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots\right)$

$= \frac{1}{2!} + \frac{2x}{3!} + \frac{3x^2}{4!} + \cdots + \frac{nx^{n-1}}{(n+1)!} + \cdots$

$= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)!}$

Therefore, $g'(1) = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}$, which means

$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$

26. (a) Factor out 2 and substitute t^2 for x in the Maclaurin series for $\frac{1}{1-x}$ at the end of Section 9.2.

$f(t) = \frac{2}{1-t^2}$

$= 2\left(\frac{1}{1-t^2}\right)$

$= 2\left[1 + t^2 + (t^2)^2 + (t^2)^3 + \cdots + (t^2)^n + \cdots\right]$

$= 2 + 2t^2 + 2t^4 + 2t^6 + \cdots + 2t^{2n} + \cdots$

(b) Since $G(0) = 0$, the constant term is zero and we may find $G(x)$ by integrating the terms of the series for $f(x)$.

$G(x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \cdots + \frac{2x^{2n+1}}{2n+1} + \cdots$

27. (a) $f(0) = (1+x)^{1/2}\Big|_{x=0} = 1$

$f'(0) = \frac{1}{2}(1+x)^{-1/2}\Big|_{x=0} = \frac{1}{2}$

$f''(0) = -\frac{1}{4}(1+x)^{-3/2}\Big|_{x=0} = -\frac{1}{4}$, so $\frac{f''(0)}{2!} = -\frac{1}{8}$

$f'''(0) = \frac{3}{8}(1+x)^{-5/2}\Big|_{x=0} = \frac{3}{8}$, so $\frac{f'''(0)}{3!} = \frac{1}{16}$

$P_4(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$

(b) Since $g(x) = f(x^2)$, the first four terms are

$1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16}.$

(c) Since $h(0) = 5$, the constant term is 5. The next three terms are obtained by integrating the first three terms of the answer to part (b). The first four terms of the series for $h(x)$ are $5 + x + \frac{x^3}{6} - \frac{x^5}{40}$.

28. (a) $a_0 = 1$

$a_1 = \frac{3}{1}a_0 = 3 \cdot 1 = 3$

$a_2 = \frac{3}{2}a_1 = \frac{3}{2} \cdot 3 = \frac{9}{2}$

$a_3 = \frac{3}{3}a_2 = a_2 = \frac{9}{2}$

Since each term is obtained by multiplying the previous term by $\frac{3}{n}$, $a_n = \frac{3^n}{n!}$.

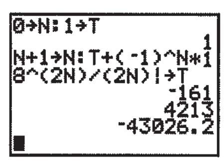
$\sum_{n=0}^{\infty} a_n x^n = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \cdots + \frac{3^n}{n!}x^n + \cdots$

(b) Since the series can be written as $\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$, it represents the function $f(x) = e^{3x}$.

(c) $f'(1) = 3e^{3x}\Big|_{x=1} = 3e^3$

29. First, note that $\cos 18 \approx 0.6603$.

Using $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, enter the following two-step commands on your home screen and continue to hit ENTER.



The sum corresponding to $N = 25$ is about 0.6582 (not within 0.001 of the exact value), and the sum corresponding to $N = 26$ is about 0.6606, which is within 0.001 of the exact value. Since we began with $N = 0$, it takes a total of 27 terms (or, up to and including the 52nd degree term).

30. One possible answer: Because the end behavior of a polynomial must be unbounded and $\sin x$ is not unbounded. Another: Because $\sin x$ has an infinite number of local extrema, but a polynomial can have only a finite number.

31. (1) Since $\sin 0 = 0$, the series for $\sin x$ is the series that begins with x. Also, $\sin x$ is an odd function, so the series for $\sin x$ will contain only odd powers of x.

(2) Since $\cos 0 = 1$, the series for $\cos x$ is the series that begins with 1. Also, $\cos x$ is an even function, so the series for $\cos x$ will contain only even powers of x.

32. Replace x by $3x$ in series for $\sin x$. Therefore, we have

$$\frac{(3x)^5}{5!} \text{ so } \frac{3^5}{5!} = \frac{81}{40}.$$

33. $$\frac{d^3}{dx^3}(\ln x)\bigg|_{x=2} = 2x^{-3}\Big|_{x=2} = \frac{1}{4}$$

Therefore, the coefficient of $(x-2)^3$ is $\frac{1/4}{3!} = \frac{1}{24}$.

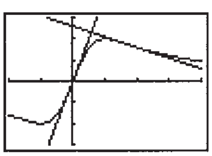
34. The linearization of f at a is the first order Taylor polynomial generated by f at $x = a$.

35. **(a)** Since $$f'(x) = \frac{d}{dx}\left(\frac{4x}{x^2+1}\right) = \frac{(x^2+1)(4)-(4x)(2x)}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2},$$

we have $f(0) = 0$, $f'(0) = 4$, $f(\sqrt{3}) = \sqrt{3}$ and

$f'(\sqrt{3}) = -\frac{1}{2}$, so the linearizations are $L_1(x) = 4x$ and

$L_2(x) = \sqrt{3} - \frac{1}{2}(x - \sqrt{3}) = -\frac{1}{2}x + \frac{3}{2}\sqrt{3}$, respectively.



[–2, 4] by [–3, 3]

(b) $f''(a)$ must be 0 because of the inflection point, so the second degree term in the Taylor series of f at $x = a$ is zero.

36. The series represents $\tan^{-1} x$. When $x = 1$, it converges to

$\tan^{-1} 1 = \frac{\pi}{4}$. When $x = -1$, it converges to

$\tan^{-1}(-1) = -\frac{\pi}{4}$.

37. True. Since the series has no constant term, $f(0)$ must be 0.

38. False. The coefficient of x^3 is $-\frac{1}{3}$

$$\frac{f'''(0)}{3!} = -\frac{1}{3}$$
$$f'''(0) = -\frac{3!}{3} = -2.$$

39. E. $f(x) = 0 + x + 0 + \frac{2x^3}{3!}$

$$f(x) = \frac{1}{3}x^3 + x$$

40. A. $(-1)^2 \frac{3^4}{4!} = \frac{27}{81}$

41. C.
$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$
$$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$
$$f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$
$$f'''\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$$
$$f^{(4)}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$
$$P_4(x) = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4$$

42. A. From the calculations above, we see that all the odd derivatives are zero. The second derivative is –1, which implies that the 6th, 10th, 14th . . . derivatives are also –1. The 4th derivative is 1, which implies that the 8th, 12th, 16th . . . derivatives are also 1.

Thus, the coefficient of $\left(x - \frac{\pi}{2}\right)^{2n}$ will be

$$\frac{f^{(2n)}\left(\frac{\pi}{2}\right)}{(2n)!} = \frac{(-1)^n}{(2n)!}.$$

The Taylor series is: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}$.

43. **(a)** $$f(x) = \frac{1}{x}(\sin x) = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots$$

(b) Because f is undefined at $x = 0$.

(c) $k = 1$

44. Note that the Maclaurin series for $\dfrac{1}{1-x}$ is $1+x+x^2+\cdots+x^n+\cdots$. If we differentiate this series and multiply by x, we obtain the desired Maclaurin series $x+2x^2+3x^3+\cdots+nx^n+\cdots$. Therefore, the desired function is $f(x)=x\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)=x\dfrac{1}{(1-x)^2}=\dfrac{x}{(x-1)^2}$.

45. (a) $f(x)=(1+x)^m$

$f'(x)=m(1+x)^{m-1}$

$f''(x)=m(m-1)(1+x)^{m-2}$

$f'''(x)=m(m-1)(m-2)(1+x)^{m-3}$

(b) Differentiating $f(x)$ k times gives $f^{(k)}(x)=m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}$. Substituting 0 for x, we have $f^{(k)}(0)=m(m-1)(m-2)\cdots(m-k+1)$.

(c) The coefficient is

$$\frac{f^{(k)}(0)}{k!}=\frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$$

(d) $f(0)=1, f'(0)=m,$ and we're done by part (c).

46. Because $f(x)=(1+x)^m$ is a polynomial of degree m. Alternately, observe that $f^{(k)}(0)=0$ for $k\geq m+1$.

Section 9.3 Taylor's Theorem (pp. 495–502)

Exploration 1 Your Turn

1. We need to consider what happens to $R_n(x)$ as $n\to\infty$. By Taylor's Theorem, $R_n(x)=\dfrac{f^{(n+1)}(c)}{(n+1)!}(x-0)^n$, where $f^{(n+1)}(c)$ is the $(n+1)$st derivative of $\cos x$ evaluated at some c between x and 0. Depending on n, the $(n+1)$st derivative of $\cos x$ is either $\cos x$, $-\sin x$, $-\cos x$, or $\sin x$. Thus for all n and for all c, $-1\leq f^{(n+1)}(c)\leq 1$. Therefore, no matter what x is, we have

$$|R_n(x)|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^n\right|\leq\left|\frac{1}{(n+1)!}x^n\right|=\frac{|x|^n}{(n+1)!}.$$ As noted in Example 3, the factorial growth in the denominator eventually outstrips the power growth in the numerator, and we have $\dfrac{|x|^n}{(n+1)!}\to 0$ for all x. This means that $R_n(x)\to 0$ for all x, which completes the proof.

Exploration 2 Euler's Formula

1. $$e^{ix}=1+ix+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\cdots+\frac{(ix)^n}{n!}+\cdots$$
$$=1+ix-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}-\cdots+(i)^n\frac{x^n}{n!}+\cdots$$

2. If we isolate the terms in the series that have i as a factor, we get:

$$e^{ix}=1+ix-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}-\cdots+(i)^n\frac{x^n}{n!}+\cdots$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\left(\frac{x^{2n+1}}{(2n+1)!}\right)+\cdots\right)$$
$$=\cos x+i\sin x.$$

(We are assuming here that we can rearrange the terms of a convergent series without affecting the sum. It happens to be true in this case, but we will see in Section 9.5 that it is not always true.)

3. $e^{i\pi}=\cos\pi+i\sin\pi=-1+0=-1$

Thus, $e^{i\pi}+1=0$

Quick Review 9.3

1. Since $|f(x)|=|2\cos(3x)|\leq 2$ on $[-2\pi, 2\pi]$ and $f(0)=2$, $M=2$.

2. Since $f(x)$ is increasing and positive on $[1, 2]$, $M=f(2)=7$.

3. Since $f(x)$ is increasing and positive on $[-3, 0]$, $M=f(0)=1$.

4. Since the minimum value of $f(x)$ is $f(-1)=-\dfrac{1}{2}$ and the maximum value of $f(x)$ is $f(1)=\dfrac{1}{2}$, $M=\dfrac{1}{2}$.

5. On $[-3, 1]$, the minimum value of $f(x)$ is $f(-3)=-7$ and the maximum value of $f(x)$ is $f(0)=2$. On $(1, 3]$, f is increasing and positive, so the maximum value of f is $f(3)=5$. Thus $|f(x)|\leq 7$ on $[-3, 3]$ and $M=7$.