Section 9.1 Exercises

1. (a) Let u_n represent the value of $*$ in the nth-term, starting with $n = 1$. Then $\frac{1}{u_1} = 1, -\frac{1}{u_2} = -\frac{1}{4}, \frac{1}{u_3} = \frac{1}{9}$, and $-\frac{1}{u_4} = -\frac{1}{16}$, so $u_1 = 1, u_2 = 4, u_3 = 9$, and $u_4 = 16$. We may write $u_n = n^2$, or $* = n^2$.

(b) Let u_n represent the value of $*$ in the nth-term, starting with $n = 0$. Then $\frac{1}{u_0} = 1, -\frac{1}{u_1} = -\frac{1}{4}, \frac{1}{u_2} = \frac{1}{9}$, and $-\frac{1}{u_3} = -\frac{1}{16}$, so $u_0 = 1, u_1 = 4, u_2 = 9$, and $u_3 = 16$. We may write $u_n = (n+1)^2$, or $* = (n+1)^2$.

(c) If $* = 3$, the series is

$(-1)^3\left(\frac{-1}{1^2}\right) + (-1)^4\left(\frac{-1}{2^2}\right) + (-1)^5\left(\frac{-1}{3^2}\right) +$

$(-1)^6\left(\frac{-1}{4^2}\right) + \cdots = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, which is the same as the desired series. Thus let $* = 3$.

2. (a) Note that $a_0 = 1$, $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{9}$, and so on. Thus

$$a_n = \left(\frac{1}{3}\right)^n.$$

(b) Note that $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$, and so on. Thus

$$a_n = \frac{(-1)^{n-1}}{n}.$$

(c) Note that $a_0 = 5$, $a_1 = 0.5$, $a_2 = 0.05$, and so on. Thus

$$a_n = 5(0.1)^n = \frac{5}{10^n}.$$

3. Different, since the terms of $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1}$ alternate between positive and negative, while the terms of $\sum_{n=1}^{\infty} -\left(\frac{1}{2}\right)^{n-1}$ are all negative.

4. The same, since both series can be represented as $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$.

5. The same, since both series can be represented as $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$.

6. Different, since $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

but $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n-1}} = -1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots$.

7.

n	term	Partial sum
1	$a_1 = 1$	$S_1 = 1$
2	$a_2 = 1.1$	$S_2 = 1 + 1.1 = 2.1$
3	$a_3 = 1.11$	$S_3 = 2.1 + 1.11 = 3.21$
4	$a_4 = 1.111$	$S_4 = 3.21 + 1.111 = 4.321$

We see that $S_1 > 1$

$S_2 > 2$

$S_3 > 3$

$\vdots$

$S_n > n$

So $\lim_{n\to\infty} S_n \ge \lim_{n\to\infty} n = \infty$

The series diverges.

8.

n	term	Partial sum
1	$a_1 = 2$	$S_1 = 2$
2	$a_2 = -1$	$S_2 = 2 - 1 = 1$
3	$a_3 = 1$	$S_3 = 1 + 1 = 2$
4	$a_4 = -1$	$S_4 = 2 - 1 = 1$
5	$a_5 = 1$	$S_5 = 1 + 1 = 2$

We see that the sequence of partial sums is 2, 1, 2, 1, 2, 1 · · ·, which has no limit. Since the sequence of partial sums has no limit, the series diverges.

9.

n	term	Partial sum
1	$a_1 = \frac{1}{2}$	$S_1 = \frac{1}{2} = 1 - \frac{1}{2}$
2	$a_2 = \frac{1}{4}$	$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4}$
3	$a_3 = \frac{1}{8}$	$S_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8}$

We see that $S_1 = 1 - \left(\frac{1}{2}\right)^1$

$S_2 = 1 - \left(\frac{1}{2}\right)^2$

$S_3 = 1 - \left(\frac{1}{2}\right)^3$

$\vdots$

$S_k = 1 - \left(\frac{1}{2}\right)^k$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1 - \left(\frac{1}{2}\right)^n\right] = 1 - 0 = 1$$

Since the sequence of partial sums converges the series converges. (The sum of the series is 1.)

10. The partial sums are 3, 3.5, 3.55, 3.555, etc. The limit of the partial sums is $3.\bar{5}$, or $3\frac{5}{9}$. The series converges.

11. Geometric series with $r=\frac{2}{3}, a=\left(\frac{2}{3}\right)^0=1$.

Converges: $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n=\frac{1}{1-\frac{2}{3}}=3$

12. Diverges, because the terms do not approach zero.

13. Geometric series with $r=\frac{2}{3}, a=\frac{5}{4}\left(\frac{2}{3}\right)^0=\frac{5}{4}$.

Converges: $\sum_{n=0}^{\infty}\left(\frac{5}{4}\right)\left(\frac{2}{3}\right)^n=\frac{\frac{5}{4}}{1-\frac{2}{3}}=\frac{15}{4}$

14. Geometric series with $r=\frac{5}{4}>1$. Diverges.

15. Diverges, because the terms alternate between 1 and -1 and do not approach zero.

16. Geometric series with $r=-0.1$ and $a=3(-0.1)^0=3$.

Converges: $\sum_{n=0}^{\infty}3(-0.1)^n=\frac{3}{1-(-0.1)}=\frac{3}{1.1}=\frac{3}{11/10}=\frac{30}{11}$.

17. Geometric series with $r=-\frac{1}{\sqrt{2}}$ and $a=1$.

Converges, since $|r|=\frac{1}{\sqrt{2}}=0.707<1$:

$$\sum_{n=0}^{\infty}\sin^n\left(\frac{\pi}{4}+n\pi\right)=1+\left(-\frac{1}{\sqrt{2}}\right)^1+\left(\frac{1}{\sqrt{2}}\right)^2+\left(-\frac{1}{\sqrt{2}}\right)^3+\cdots$$
$$=\sum_{n=0}^{\infty}\left(-\frac{1}{\sqrt{2}}\right)^n=\frac{1}{1-\left(-\frac{1}{\sqrt{2}}\right)}=\frac{\sqrt{2}}{\sqrt{2}+1}$$
$$=\frac{\sqrt{2}\left(\sqrt{2}-1\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}=\frac{2-\sqrt{2}}{2-1}=2-\sqrt{2}$$

18. $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0$

Since the sequence of terms does not approach 0, the series diverges.

19. Geometric series with $r=\frac{e}{\pi}\approx 0.865<1$, and $a=\left(\frac{e}{\pi}\right)^1$.

Converges: $\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n=\frac{\frac{e}{\pi}}{1-\frac{e}{\pi}}=\frac{e}{\pi-e}$.

20. Geometric series with $r=\frac{5}{6}$ and $a=\frac{1}{6}\left(\frac{5}{6}\right)^0=\frac{1}{6}$.

Converges: $\sum_{n=0}^{\infty}\frac{5^n}{6^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^n=\frac{\frac{1}{6}}{1-\frac{5}{6}}=1$.

21. Since $\sum_{n=0}^{\infty}2^n x^n=\sum_{n=0}^{\infty}(2x)^n$, the series converges when $|2x|<1$ and the interval of convergence is $\left(-\frac{1}{2},\frac{1}{2}\right)$. Since the sum of the series is $\frac{1}{1-2x}$, the series represents the function $f(x)=\frac{1}{1-2x}$, $-\frac{1}{2}<x<\frac{1}{2}$.

22. Since $\sum_{n=0}^{\infty}(-1)^n(x+1)^n=\sum_{n=0}^{\infty}[-(x+1)]^n$, the series converges when $|-(x+1)|<1$ and the interval of convergence is $(-2, 0)$. Since the sum of the series is $\frac{1}{1-[-(x+1)]}=\frac{1}{x+2}$, the series represents the function $f(x)=\frac{1}{x+2}, -2<x<0$.

23. Since $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(x-3)^n=\sum_{n=0}^{\infty}\left(\frac{3-x}{2}\right)^n$, the series converges when $\left|\frac{3-x}{2}\right|<1$ and the interval of convergence is $(1, 5)$. Since the sum of the series is $\frac{1}{1-\frac{(3-x)}{2}}=\frac{2}{x-1}$, the series represents the function $f(x)=\frac{2}{x-1}, 1<x<5$.

24. For $\sum_{n=0}^{\infty}3\left(\frac{x-1}{2}\right)^n$, the series converges when $\left|\frac{x-1}{2}\right|<1$ and the interval of convergence is $(-1, 3)$. Since the sum of the series is $\frac{3}{1-\frac{(x-1)}{2}}=\frac{6}{3-x}$, the series represents the function $f(x)=\frac{6}{3-x}, -1<x<3$.

25. Since $\sum_{n=0}^{\infty} \sin^n x = \sum_{n=0}^{\infty} (\sin x)^n$, the series converges when $|\sin x| < 1$. Thus, the series converges for all values of x except odd integer multiples of $\frac{\pi}{2}$, that is, $x \neq (2k+1)\frac{\pi}{2}$ for integers k. Since the sum of the series is $\frac{1}{1-\sin x}$, the series represents the function

$$f(x) = \frac{1}{1-\sin x},\ x \neq (2k+1)\frac{\pi}{2}.$$

26. Since $\sum_{n=0}^{\infty} \tan^n x = \sum_{n=0}^{\infty} (\tan x)^n$, the series converges when $|\tan x| < 1$. Thus, the series converges for $-\frac{\pi}{4} + k\pi < x < \frac{\pi}{4} + k\pi$, where k is an integer. Since the sum of the series is $\frac{1}{1-\tan x}$, the series represents the function

$$f(x) = \frac{1}{1-\tan x},\ -\frac{\pi}{4} + k\pi < x < \frac{\pi}{4} + k\pi.$$

27.
$$f(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n,\ -\frac{1}{2} < x < \frac{1}{2}$$
$$\frac{d}{x}\left(\frac{1}{1-2x}\right) = \frac{d}{dx}\left(1 + 2x + 4x^2 + 8x^3 + \cdots + 2^n x^n + \cdots\right)$$
$$\frac{2}{(1-2x)^2} = 2 + 8x + 24x^2 + \cdots + 2^n nx^{n-1} + \cdots$$
$$f'(x) = \frac{2}{(2x-1)^2} = \sum_{n=1}^{\infty} 2^n nx^{n-1},\ -\frac{1}{2} < x < \frac{1}{2}$$

28.
$$f(x) = \frac{1}{x+2} = \sum_{n=0}^{\infty} (-1)^n (x+1)^n,\ -2 < x < 0$$
$$\frac{d}{dx}\left(\frac{1}{x+2}\right) = \frac{d}{dx}\left(1 - (x+1) + (x+1)^2 - (x+1)^3 + \cdots + (-1)^n (x+1)^n + \cdots\right)$$
$$-\frac{1}{(x+2)^2} = -1 + 2(x+1) - 3(x+1)^2 + \cdots + (-1)^n n(x+1)^{n-1} + \cdots$$
$$f'(x) = -\frac{1}{(x+2)^2} = \sum_{n=1}^{\infty} (-1)^n n(x+1)^{n-1},\ -2 < x < 0$$

29.
$$f(x) = \frac{2}{x-1} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n,\ 1 < x < 5$$
$$\frac{d}{dx}\left(\frac{2}{x-1}\right) = \frac{d}{dx}\left(1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots\right)$$
$$-\frac{2}{(x-1)^2} = -\frac{1}{2} + \frac{1}{2}(x-3) - \frac{3}{8}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n n(x-3)^{n-1} + \cdots$$
$$f'(x) = -\frac{2}{(x-1)^2} = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n n(x-3)^{n-1},\ 1 < x < 5$$

30.
$$f(x) = \frac{6}{3-x} = \sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n,\ -1 < x < 3$$
$$\frac{d}{dx}\left(\frac{6}{3-x}\right) = \frac{d}{dx}\left(3 + 3\left(\frac{x-1}{2}\right) + 3\left(\frac{x-1}{2}\right)^2 + \cdots + 3\left(\frac{x-1}{2}\right)^n + \cdots\right)$$
$$\frac{6}{(3-x)^2} = \frac{3}{2} + 3\left(\frac{x-1}{2}\right) + \cdots + \frac{3}{2}n\left(\frac{x-1}{2}\right)^{n-1} + \cdots$$
$$f'(x) = \frac{6}{(x-3)^2} = \sum_{n=1}^{\infty} \frac{3}{2}n\left(\frac{x-1}{2}\right)^{n-1},\ -1 < x < 3$$

31. $f(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n,\ -\frac{1}{2} < x < \frac{1}{2}$

The series is centered at 0, that is, $a = 0$. We need to find the power series for $\int_0^x f(t)\,dt$.

$$\int_0^x \frac{1}{1-2t}\,dt = \int_0^x (1 + 2t + 4t^2 + 8t^3 + \cdots + 2^n t^n + \cdots)\,dt$$
$$-\frac{1}{2}\ln|1-2t|\Big]_0^x = t + t^2 + \frac{4}{3}t^3 + 2t^4 + \cdots + \frac{2^n}{n+1}t^{n+1}\Big]_0^x$$
$$-\frac{1}{2}\ln|1-2x| = x + x^2 + \frac{4}{3}x^3 + 2x^4 + \cdots + \frac{2^n}{n+1}x^{n+1}$$
$$-\frac{1}{2}\ln|2x-1| = \sum_{n=0}^{\infty} \frac{2^n}{n+1}x^{n+1},\ -\frac{1}{2} < x < \frac{1}{2}$$

32. $f(x)=\dfrac{1}{x+2}=\sum_{n=0}^{\infty}(-1)^n(x+1)^n,\ -2<x<0$

The power series is centered at -1, that is, $a=-1$. We need to find the power series for $\int_{-1}^{x} f(t)\,dt$:

$$\int_{-1}^{x}\frac{1}{t+2}dt=\int_{-1}^{x}(1-(t+1)+(t+1)^2-(t+1)^3+\cdots+(-1)^n(t+1)^n+\cdots)dt$$

$$\ln|t+2|\Big]_{-1}^{x}=\left[t-\frac{1}{2}(t+1)^2+\frac{1}{3}(t+1)^3-\frac{1}{4}(t+1)^4+\cdots+\frac{(-1)^n}{n+1}(t+1)^{n+1}+\cdots\right]_{-1}^{x}$$

$$\ln|x+2|=x-(-1)-\frac{1}{2}(x+1)^2+\frac{1}{3}(x+1)^3-\frac{1}{4}(x+1)^4+\cdots+\frac{(-1)^n}{n+1}(x+1)^{n+1}+\cdots$$

$$\ln|x+2|=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x+1)^{n+1}$$

33. $f(x)=\dfrac{2}{x-1}=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(x-3)^n,\ 1<x<5$

The power series is centered at 3, that is, $a=3$. We need to find the power series for $\int_{3}^{x} f(t)\,dt$:

$$\int_{3}^{x}\frac{2}{t-1}dt=\int_{3}^{x}\left(1-\frac{1}{2}(t-3)+\frac{1}{4}(t-3)^2-\frac{1}{8}(t-3)^3+\cdots+\left(-\frac{1}{2}\right)^n(t-3)^n\cdots\right)dt$$

$$2\ln|t-1|\Big]_{3}^{x}=\left[t-\frac{1}{4}(t-3)^2+\frac{1}{12}(t-3)^3-\frac{1}{32}(t-3)^4+\cdots+\left(-\frac{1}{2}\right)^n\frac{(t-3)^{n+1}}{n+1}+\cdots\right]_{3}^{x}$$

$$2\ln|x-1|-2\ln 2=x-3-\frac{1}{4}(x-3)^2+\frac{1}{12}(x-3)^3-\frac{1}{32}(x-3)^4+\cdots+\left(-\frac{1}{2}\right)^n\frac{(x-3)^{n+1}}{n+1}+\cdots$$

$$2\ln\left|\frac{x-1}{2}\right|=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{(x-3)^{n+1}}{n+1},\ 1<x<5$$

34. $f(x)=\dfrac{6}{3-x}=\sum_{n=0}^{\infty}3\left(\frac{x-1}{2}\right)^n,\ -1<x<3$

The power series is centered at 1, that is, $a=1$. We need to find the power series for $\int_{1}^{x} f(t)dt$:

$$\int_{1}^{x}\frac{6}{3-t}dt=\int_{1}^{x}\left(3+3\left(\frac{t-1}{2}\right)+3\left(\frac{t-1}{2}\right)^2+\cdots+3\left(\frac{t-1}{2}\right)^n+\cdots\right)dt$$

$$-6\ln|3-t|\Big]_{1}^{x}=\left[3t+3\left(\frac{t-1}{2}\right)^2+2\left(\frac{t-1}{2}\right)^3+\cdots+\frac{6}{n+1}\left(\frac{t-1}{2}\right)^{n+1}+\cdots\right]_{1}^{x}$$

$$-6\ln|3-x|+6\ln 2=3x-3+3\left(\frac{x-1}{2}\right)^2+2\left(\frac{x-1}{2}\right)^3+\cdots+\frac{6}{n+1}\left(\frac{x-1}{2}\right)^{n+1}+\cdots$$

$$-6\ln\left|\frac{3-x}{2}\right|=\sum_{n=0}^{\infty}\frac{6}{n+1}\left(\frac{x-1}{2}\right)^{n+1},\ -1<x<3$$

or

$$6\ln\left(\frac{2}{|x-3|}\right)=\sum_{n=0}^{\infty}\frac{6}{n+1}\left(\frac{x-1}{2}\right)^{n+1},\ -1<x<3$$

35. **(a)** Since the terms are all positive and do not approach zero, the partial sums tend toward infinity.

(b) The partial sums are alternately 1 and 0.

(c) The partial sums alternate between positive and negative while their magnitude increases toward infinity.

36. Since $\sum_{n=0}^{\infty}\dfrac{e^{n\pi}}{\pi^{ne}}=\sum_{n=0}^{\infty}\left(\dfrac{e^{\pi}}{\pi^{e}}\right)^n$, this is a geometric series with common ratio $r=\dfrac{e^{\pi}}{\pi^{e}}\approx 1.03$, which is greater than one.

37. $\sum_{n=0}^{\infty}x^n=20$

$$\frac{1}{1-x}=20,\ |x|<1$$
$$1=20-20x$$
$$20x=19$$
$$x=\frac{19}{20}$$

38. One possible answer:
For any real number $a \neq 0$,

use $\frac{a}{2}+\frac{a}{4}+\frac{a}{8}+\frac{a}{16}+\frac{a}{32}+\cdots$. To get 0,

use $1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}-\frac{1}{32}-\cdots$.

39. Assuming the series begins at $n = 1$;

(a) $$\sum_{n=1}^{\infty} 2r^{n-1} = \frac{2}{1-r} = 5,\ |r|<1$$
$$2 = 5-5r$$
$$5r = 3$$
$$r = \frac{3}{5}$$

Series: $\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^{n-1}$

(b) $$\sum_{n=1}^{\infty} \frac{13}{2} r^{n-1} = \frac{\frac{13}{2}}{1-r} = 5,\ |r|<1$$
$$\frac{13}{2} = 5-5r$$
$$5r = -\frac{3}{2}$$
$$r = -\frac{3}{10}$$

Series: $\sum_{n=1}^{\infty} \frac{13}{2}\left(-\frac{3}{10}\right)^{n-1}$

40. Let $a = \frac{21}{100}$ and $r = \frac{1}{100}$, giving
$$0.\overline{21} = 0.21+0.21(0.01)+0.21(0.01)^2 +0.21(0.01)^3+\cdots$$
$$= \sum_{n=0}^{\infty} 0.21(0.01)^n$$
$$= \frac{0.21}{1-0.01}$$
$$= \frac{0.21}{0.99}$$
$$= \frac{7}{33}$$

41. Let $a = \frac{234}{1000}$ and $r = \frac{1}{1000}$, giving
$$0.\overline{234} = 0.234+0.234(0.001)+0.234(0.001)^2 +0.234(0.001)^3+\cdots$$
$$= \sum_{n=0}^{\infty} 0.234(0.001)^n$$
$$= \frac{0.234}{1-0.001}$$
$$= \frac{0.234}{0.999}$$
$$= \frac{26}{111}$$

42. $$0.\overline{7} = 0.7+0.7(0.1)+0.7(0.1)^2+0.7(0.1)^3+\cdots$$
$$= \sum_{n=0}^{\infty} 0.7(0.1)^n$$
$$= \frac{0.7}{1-0.1}$$
$$= \frac{0.7}{0.9}$$
$$= \frac{7}{9}$$

43. $$0.\overline{d} = \frac{d}{10}+\frac{d}{10}(0.1)+\frac{d}{10}(0.1)^2+\frac{d}{10}(0.1)^3+\cdots$$
$$= \sum_{n=0}^{\infty} \frac{d}{10}(0.1)^n$$
$$= \frac{\frac{d}{10}}{1-0.1}$$
$$= \frac{d}{10-1}$$
$$= \frac{d}{9}$$

44. $$0.0\overline{6} = 0.06+0.06(0.1)+0.06(0.1)^2+0.06(0.1)^3+\cdots$$
$$= \sum_{n=0}^{\infty} 0.06(0.1)^n$$
$$= \frac{0.06}{1-0.1}$$
$$= \frac{0.06}{0.9}$$
$$= \frac{1}{15}$$

45. $$1.\overline{414} = 1+0.414+0.414(0.001)+0.414(0.001)^2+\cdots$$
$$= 1+\sum_{n=0}^{\infty} 0.414(0.001)^n$$
$$= 1+\frac{0.414}{1-0.001}$$
$$= 1+\frac{46}{111}$$
$$= \frac{157}{111}$$

46. $$1.24\overline{123} = 1.24+0.00123+0.00123(0.001) +0.00123(0.001)^2+\cdots$$
$$= 1.24+\sum_{n=1}^{\infty} 0.00123(0.001)^n$$
$$= 1.24+\frac{0.00123}{0.999}$$
$$= \frac{124}{100}+\frac{41}{33{,}300}$$
$$= \frac{41{,}333}{33{,}300}$$

47. $3.\overline{142857} = 3 + 0.142857(1 + 0.000001 + 0.000001^2 + \cdots)$

$$= 3 + 0.142857 \sum_{n=0}^{\infty} 0.000001^n$$
$$= 3 + (0.142857)\left(\frac{1}{1 - 0.000001}\right)$$
$$= 3 + \frac{0.142857}{0.999999}$$
$$= 3 + \frac{1}{7}$$
$$= \frac{22}{7}$$

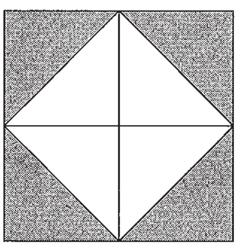
48. Total distance $= 4 + 2[4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots]$

$$= 4 + 2\sum_{n=0}^{\infty} 2.4(0.6)^n$$
$$= 4 + 2 \bullet \frac{2.4}{1 - 0.6}$$
$$= 4 + 2 \bullet 6$$
$$= 16 \text{ m}$$

49. Total time

$$= \sqrt{\frac{4}{4.9}} + 2\left[\sqrt{\frac{4(0.6)}{4.9}} + \sqrt{\frac{4(0.6)^2}{4.9}} + \sqrt{\frac{4(0.6)^3}{4.9}} + \ldots\right]$$
$$= \sqrt{\frac{4}{4.9}} + 2\sqrt{\frac{4(0.6)}{4.9}}\left[1 + \sqrt{0.6} + (\sqrt{0.6})^2 + \ldots\right]$$
$$= \sqrt{\frac{4}{4.9}} + 2\sqrt{\frac{4(0.6)}{4.9}} \bullet \frac{1}{1 - \sqrt{0.6}}$$
$$\approx 7.113 \text{ sec}$$

50.



The area of each square is half of the area of the preceding square (see diagram above), so the total of all the areas is

$$\sum_{n=0}^{\infty} 4\left(\frac{1}{2}\right)^n$$
$$= \frac{4}{1 - \left(\frac{1}{2}\right)} = 8\,\text{m}^2.$$

51. Total area $= \sum_{n=1}^{\infty} 2^n \bullet \frac{1}{2} \bullet \pi\left(\frac{1}{2^n}\right)^2$

$$= \sum_{n=1}^{\infty} \frac{\pi}{2} \bullet \left(\frac{1}{2}\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{\pi}{4}\left(\frac{1}{2}\right)^n$$
$$= \frac{\frac{\pi}{4}}{1 - \left(\frac{1}{2}\right)}$$
$$= \frac{\pi}{2}$$

52. **(a)** $S - rS = (a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1})$

$$-(ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n)$$
$$= a - ar^n$$

(b) Factor and divide by $1 - r$:

$$S - rS = a - ar^n$$
$$S(1 - r) = a - ar^n$$
$$S = \frac{a - ar^n}{1 - r}$$

53. Using the notation $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$, the formula from Exercise 52 is $S_n = \frac{a - ar^n}{1 - r}$.

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$ and so

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1 - r} = \frac{a}{1 - r}.$$

If $|r| > 1$ or $r = -1$, then r^n has no finite limit as $n \to \infty$, so the expression $\frac{a - ar^n}{1 - r}$ has no finite limit and $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

If $r = 1$, then the nth partial sum is na, which goes to $\pm\infty$, depending on the sign of a.

54. Comparing $\frac{1}{1+3x}$ with $\frac{a}{1-r}$, the leading term is $a = 1$and the common ratio is $r = -3x$.

Series: $1 - 3x + 9x^2 - \cdots + (-3x)^n + \cdots$

Interval: The series converges when$|-3x| < 1$, so the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

55. Comparing $\frac{x}{1-2x}$ with $\frac{a}{1-r}$, the first term is $a = x$ and the common ratio is $r = 2x$.

Series: $x + 2x^2 + 4x^3 + \cdots + 2^{n-1}x^n + \cdots$

Interval: The series converges when $|2x| < 1$, so the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

56. Comparing $\frac{3}{1-x^3}$ with $\frac{a}{1-r}$, the first term is $a = 3$ and the common ratio is $r = x^3$.

Series: $3 + 3x^3 + 3x^6 + \cdots + 3x^{3n} + \cdots$

Interval: The series converges when $|x^3| < 1$, so the interval of convergence is $(-1, 1)$.

57. Comparing $\frac{1}{1+(x-4)}$ with $\frac{a}{1-r}$, the first term is $a = 1$ and the common ratio is $r = -(x-4)$.

Series: $1 - (x-4) + (x-4)^2 - \cdots + (-1)^n(x-4)^n + \cdots$

Interval: The series converges when $|x-4| < 1$, so the interval of convergence is $(3, 5)$.

58. Comparing $\frac{1}{4}\left(\frac{1}{1+(x-1)}\right)$ with $\frac{a}{1-r}$, the first term is $a = \frac{1}{4}$ and the common ratio is $r = -(x-1)$.

Series: $\frac{1}{4} - \frac{1}{4}(x-1) + \frac{1}{4}(x-1)^2 - \cdots + \frac{1}{4}(-1)^n(x-1)^n + \cdots$

Interval: The series converges when $|x-1| < 1$, so the interval of convergence is $(0, 2)$.

59. Rewriting $\frac{1}{2-x}$ as $\frac{1}{1-(x-1)}$ and comparing with $\frac{a}{1-r}$, the first term is $a = 1$ and the common ratio is $r = x-1$.

Series: $1 + (x-1) + (x-1)^2 + \cdots + (x-1)^n + \cdots$

Interval: The series converges when $|x-1| < 1$, so the interval of convergence is $(0, 2)$.

Alternate solution:

Rewriting $\frac{1}{2-x}$ as $\frac{1}{2}\left(\frac{1}{1-\left(\frac{x}{2}\right)}\right)$ and comparing with $\frac{a}{1-r}$, the first is $a = \frac{1}{2}$ and the common ratio is $r = \frac{x}{2}$.

Series: $\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \cdots + \frac{1}{2^{n+1}}x^n + \cdots$

Interval: The series converges when $\left|\frac{x}{2}\right| < 1$, so the interval of convergence is $(-2, 2)$.

60.
$$\begin{aligned}
1 + e^b + e^{2b} + e^{3b} + \cdots &= \sum_{n=0}^{\infty}(e^b)^n \\
&= \frac{1}{1-e^b} = 9 \\
1 &= 9 - 9e^b \\
9e^b &= 8 \\
e^b &= \frac{8}{9} \\
b &= \ln\left(\frac{8}{9}\right)
\end{aligned}$$

61. (a) When $t = 1$, $S = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\left(\frac{1}{2}\right)} = 2$.

(b) S converges when $\left|\frac{t}{t+1}\right| < 1$

Equivalently: $\left|\frac{t+1}{t}\right| > 1$

$\left|1 + \frac{1}{t}\right| > 1$

$1 + \frac{1}{t} < -1$ OR $1 + \frac{1}{t} > 1$

$\frac{1}{t} < -2$ $\qquad$ $\frac{1}{t} > 0$

$t > -\frac{1}{2}$ $\qquad$ $t > 0$

Thus, S converges for all $t > -\frac{1}{2}$.

(c) For $t > -\frac{1}{2}$, we have

$$S = \sum_{n=0}^{\infty}\left(\frac{t}{1+t}\right)^n = \frac{1}{1-\frac{t}{1+t}} = \frac{1+t}{(1+t)-t} = 1+t, \text{ so } S > 10$$

when $t > 9$.

62. (a) The series converges to S means that $\lim_{n\to\infty} S_n = S$, where $S_n = \sum_{k=1}^{k=n} a_k$ is the nth partial sum of the series.

(b) $S_n = (a_1 + \cdots + a_{n-1}) + a_n = S_{n-1} + a_n$

(c) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n$ so $S = S + \lim_{n\to\infty} a_n$ or $\lim_{n\to\infty} a_n = 0$

63. Since $\frac{1}{x}=1-(x-1)+(x-1)^2-(x-1)^3+\cdots$
$+(-1)^n(x-1)^n+\cdots$, we may write:
$$\ln x=\int_1^x \frac{1}{t}\,dt=x-1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^n(x-1)^{n+1}}{n+1}$$

64. To determine our starting point, we note that $\int f(x)dx=\int 2(1-x)^{-3}dx=(1-x)^{-2}+C$. Using the result from Example 4, we have:
$$(1-x)^{-2}=1+2x+3x^2+4x^3+\cdots+nx^{n-1}+\cdots$$
$$\frac{d}{dx}(1-x)^{-2}=\frac{d}{dx}(1+2x+3x^2+4x^3+\cdots+nx^{n-1}+\cdots)$$
$$2(1-x)^{-3}=0+2+6x+12x^2+\cdots+n(n-1)x^{n-2}+\cdots$$
Thus, $f(x)=2+6x+12x^2+\cdots+n(n-1)x^{n-2}+\cdots$.
Replacing n by $n+2$, this may be written as
$f(x)=2+6x+12x^2+(n+2)(n+1)x^n+\cdots$.
Interval: The series converges when $|x|<1$, so the interval of convergence is $(-1, 1)$.

65. (a) No, because if you differentiate it again, you would have the original series for f, but by Theorem 1, that would have to converge for $-2<x<2$, which contradicts the assumption that the original series converges only for $-1<x<1$.

(b) No, because if you integrate it again, you would have the original series for f, but by Theorem 2, that would have to converge for $-2<x<2$, which contradicts the assumption that the original series converges only for $-1<x<1$.

66. False. It diverges because it is a geometric series with ratio $r=1.01>1$.

67. False. It converges because it is a geometric series with ratio $r=\frac{1}{2}<1$.

68. C. $\frac{1}{1-1/3}=\frac{1}{2/3}=\frac{3}{2}$

69. A. Converges for $|x-1|<1$, or $0<x<2$.

70. E. $\frac{1}{1-(x-1)}=\frac{1}{2-x}$

71. D. The function must be the antiderivative of $\frac{1}{2-x}$ that passes through the point $(1, 0)$, and that function is $-\ln(2-x)$.

72. (a) Comparing $f(t)=\frac{4}{1+t^2}$ with $\frac{a}{1-r}$, the first term is $a=4$ and the common ratio is $r=-t^2$.
First four terms: $4-4t^2+4t^4-4t^6$
General term: $(-1)^n(4t^{2n})$

(b) Note that $G(0)=0$, so the constant term of the power series for $G(x)$ will be 0. Integrate the terms for $f(x)$ to obtain the terms for $G(x)$.
First four terms: $4x-\frac{4}{3}x^3+\frac{4}{5}x^5-\frac{4}{7}x^7$
General term: $(-1)^n\left(\frac{4}{2n+1}\right)x^{2n+1}$

(c) The series in part (a) converges when $|-t^2|<1$, so the interval of convergence is $(-1, 1)$.

(d) The two numbers are $x=\pm 1$, which result in the convergent series
$$G(1)=4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\cdots+(-1)^n\left(\frac{4}{2n+1}\right)+\cdots \text{ and}$$
$$G(-1)=-4+\frac{4}{3}-\frac{4}{5}+\frac{4}{7}-\cdots+(-1)^{n-1}\left(\frac{4}{2n+1}\right)+\cdots,$$
respectively.

73. Let $L=\lim_{n\to\infty}a_n$. Then by definition of convergence, for $\frac{\epsilon}{2}$ there corresponds an N such that for all m and n,
$n,m>N \Rightarrow |a_m-L|<\frac{\epsilon}{2}$ and $|a_n-L|<\frac{\epsilon}{2}$. Now,
$$|a_m-a_n|=|a_m-L+L-a_n|\le|a_m-L|+|a_n-L|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$
whenever $m>N$ and $n>N$.

74. Given an $\epsilon>0$, by definition of convergence there corresponds an N such that for all $n<N, |L_1-a_n|<\epsilon$ and $|L_2-a_n|<\epsilon$. (There is one such number for each series, and we may let N be the larger of the two numbers.) Now
$$\begin{aligned}|L_2-L_1|&=|L_2-a_n+a_n-L_1|\\&\le|L_2-a_n|+|a_n-L_n|\\&<\epsilon+\epsilon\\&=2\epsilon\end{aligned}$$
$|L_2-L_1|<2\epsilon$ says that the difference between two fixed values is smaller than any positive number 2ϵ. The only nonnegative number smaller than every positive number is 0, so $|L_2-L_1|=0$ or $L_1=L_2$.

75. Consider the two subsequences $a_{k(n)}$ and $a_{i(n)}$, where $\lim_{n\to\infty} a_{k(n)} = L_1$, $\lim_{n\to\infty} a_{i(n)} = L_2$, and $L_1 \neq L_2$. Given an $\epsilon > 0$ there corresponds an N_1 such that for $k(n) > N_1$, $|a_{k(n)} - L_1| < \epsilon$, and an N_2 such that for $i(n) > N_2$, $|a_{i(n)} - L_2| < \epsilon$. Assume a_n converges.

Let $N = \max\{N_1, N_2\}$. Then for $n > N$, we have that $|a_n - L_1| < \epsilon$ and $|a_n - L_2| < \epsilon$ for infinitely many n. This implies that $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$ where $L_1 \neq L_2$.

Since the limit of a sequence is unique (by Exercise 74), a_n does not converge and hence diverges.

76. (a)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(3n+1)}{(n+1)} \bullet \frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1+\frac{1}{n}} = \frac{3+0}{1+0}$$

$$= 3$$

(b) We showed in part (a) that $\lim_{n\to\infty} a_n = 3$.

A similar calculation would show that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x+1}{x+1} = 3$$

This tells us that as x gets infinitely large, the graph $y = f(x)$ gets infinitely close to the line $y = 3$. This is exactly what it means for $y = 3$ to be a horizontal asymptote of the graph.

Section 9.2 Taylor Series (pp. 484–494)

Exploration 1 Designing a Polynomial to Specifications

1. Since $P(0) = 1$, we know that the constant coefficient is 1. Since $P'(0) = 2$, we know that the coefficient of x is 2.

Since $P''(0) = 3$, we know that the coefficient of x^2 is $\frac{3}{2}$. (The 2 in the denominator is needed to cancel the factor of 2 that results from differentiating x^2.) Similarly, we find the coefficients of x^3 and x^4 to be $\frac{4}{6}$ and $\frac{5}{24}$.

Thus, $P(x) = 1 + 2x + \frac{3}{2}x^2 + \frac{4}{6}x^3 + \frac{5}{24}x^4$.

Exploration 2 A Power Series for the Cosine

1. $\cos(0) = 1$

$\cos'(0) = -\sin(0) = 0$

$\cos''(0) = -\cos(0) = -1$

$\cos^{(3)}(0) = \sin(0) = 0$

etc.

The pattern 1, 0, −1, 0 will repeat forever. Therefore, $P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$, and the Taylor series is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots.$$

2. A clever shortcut is simply to differentiate the previously-discovered series for $\sin x$ term-by-term!

Exploration 3 Approximating sin 13

1. 0.4201670368...

4. 20 terms.

Quick Review 9.2

1. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$f^{(n)}(x) = 2^n e^{2x}$

2. $f(x) = \dfrac{1}{x-1}$

$f'(x) = -(x-1)^{-2}$

$f''(x) = 2(x-1)^{-3}$

$f'''(x) = -6(x-1)^{-4}$

$f^{(n)}(x) = (-1)^n n!(x-1)^{-(n+1)}$

3. $f(x) = 3^x$

$f'(x) = 3^x \ln 3$

$f''(x) = 3^x(\ln 3)^2$

$f'''(x) = 3^x(\ln 3)^3$

$f^{(n)}(x) = 3^x(\ln 3)^n$

4. $f(x) = \ln(x)$

$f'(x) = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$

$f^{(4)}(x) = -6x^{-4}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$ for $n \geq 1$