Section 8.4 Exercises

1. (a) $\int_0^\infty \frac{2x}{x^2+1}dx = \lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}dx$

(b) $\lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}dx = \lim_{b\to\infty}\left(\ln(x^2+1)\right)\Big|_0^b = \infty$

diverges

2. (a) $\int_1^\infty \frac{dx}{x^{1/3}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^{1/3}}$

(b) $\lim_{b\to\infty}\int_1^b \frac{dx}{x^{1/3}} = \lim_{b\to\infty}\left(\frac{3}{2}x^{2/3}\right)\Big|_1^b = \infty$

diverges

3. (a) $\int_{-\infty}^\infty \frac{2x}{(x^2+1)^2}dx = \lim_{b\to-\infty}\int_b^0 \frac{2x}{(x^2+1)^2}dx + \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}dx$

(b) $\lim_{b\to-\infty}\int_b^0 \frac{2x}{(x^2+1)^2}dx + \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}dx$

$= \lim_{b\to-\infty}\left(\frac{-1}{x^2+1}\right)\Big|_b^0 + \lim_{b\to\infty}\left(\frac{-1}{x^2+1}\right)\Big|_0^b = 0$

converges

4. (a) $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}}$

(b) $\lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}(2\sqrt{x})\Big|_1^b = \infty$

diverges

5. $\int_1^\infty \frac{dx}{x^4} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^4} = \lim_{b\to\infty}\left(\frac{-1}{3x^3}\right)\Big|_1^b = \frac{1}{3}$

6. $\int_1^\infty \frac{2dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{2dx}{x^3} = \lim_{b\to\infty}\left(\frac{-2}{2x^2}\right)\Big|_1^b = 1$

7. $\int_1^\infty \frac{dx}{\sqrt[3]{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt[3]{x}} = \lim_{b\to\infty}\left(\frac{3}{2}(x)^{2/3}\right)\Big|_1^b = \infty$

diverges

8. $\int_1^\infty \frac{dx}{\sqrt[4]{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt[4]{x}} = \lim_{b\to\infty}\left(\frac{4}{3}(x)^{3/4}\right)\Big|_1^b = \infty$

diverges

9. $\int_{-\infty}^{-1} \frac{dx}{x^2} = \lim_{b\to-\infty}\int_b^{-1} \frac{dx}{x^2} = \lim_{b\to-\infty}\left(-\frac{1}{x}\right)\Big|_b^{-1} = 1$

10. $\int_{-\infty}^0 \frac{dx}{(x-2)^3} = \lim_{b\to-\infty}\int_b^0 \frac{dx}{(x-2)^3}$

$= \lim_{b\to-\infty}\left(\frac{-1}{2(x-2)^2}\right)\Big|_b^0 = -1/8$

11. $\int_{-\infty}^{-2} \frac{2dx}{x^2-1} = \lim_{b\to-\infty}\int_b^{-2}\left[\frac{2}{x^2-1}\right]dx$

$= \lim_{b\to-\infty}\int_b^{-2}\left[\frac{1}{x-1}-\frac{1}{x+1}\right]dx = \lim_{b\to-\infty}\left(\ln\left|\frac{x-1}{x+1}\right|\right)\Big|_b^{-2}$

$= \lim_{b\to-\infty}\left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right) = \ln 3$

12. $\int_2^\infty \frac{3dx}{x^2-x} = \lim_{b\to\infty}\int_2^b\left[\frac{3}{x^2-x}\right]dx = \lim_{b\to\infty}\int_2^b\left[\frac{3}{x-1}-\frac{3}{x}\right]dx$

$= \lim_{b\to\infty}\left(3\ln\left|\frac{x-1}{x}\right|\right)\Big|_2^b$

$= \lim_{b\to\infty}\left(3\ln\left|\frac{b-1}{b}\right| - 3\ln\left(\frac{1}{2}\right)\right) = 3\ln 2$

13. $\int_{-1}^\infty \frac{dx}{x^2+5x+6} = \lim_{b\to\infty}\int_{-1}^b\left[\frac{1}{x^2+5x+6}\right]dx$

$= \lim_{b\to\infty}\int_{-1}^b\left[\frac{1}{x+2}-\frac{1}{x+3}\right]dx$

$= \lim_{b\to\infty}\left(\ln\left|\frac{x+2}{x+3}\right|\right)\Big|_{-1}^b$

$= \lim_{b\to\infty}\left(\ln\left|\frac{b+2}{b+3}\right| - \ln\left(\frac{1}{2}\right)\right) = \ln 2$

14. $\int_{-\infty}^0 \frac{2dx}{x^2-4x+3} = \lim_{b\to-\infty}\int_b^0\left[\frac{2}{x^2-4x+3}\right]dx$

$= \lim_{b\to-\infty}\int_b^0\left[\frac{1}{x-3}-\frac{1}{x-1}\right]dx$

$= \lim_{b\to-\infty}\left(\ln\left|\frac{x-3}{x-1}\right|\right)\Big|_b^0$

$= \lim_{b\to-\infty}\left(\ln 3 - \ln\left|\frac{b-3}{b-1}\right|\right) = \ln 3$

15. $\int_1^\infty \frac{5x+6}{x^2+2x}dx = \lim_{b\to\infty}\int_1^b\left[\frac{5x+6}{x^2+2x}\right]dx$

$= \lim_{b\to\infty}\int_1^b\left[\frac{3}{x}+\frac{2}{x+2}\right]dx$

$= \lim_{b\to\infty}\left(\ln\left|x^3(x+2)^2\right|\right)\Big|_1^b = \infty$

diverges

16. $\int_{-2}^{-\infty} \frac{2dx}{x^2-2x} = \lim_{b\to-\infty}\int_{-2}^b\left[\frac{2}{x^2-2x}\right]dx$

$= \lim_{b\to-\infty}\int_{-2}^b\left[\frac{1}{x-2}-\frac{1}{x}\right]dx$

$= \lim_{b\to-\infty}\left(\ln\left|\frac{x-2}{x}\right|\right)\Big|_{-2}^b$

$= \lim_{b\to-\infty}\left(\ln\left|\frac{b-2}{b}\right| - \ln 2\right) = -\ln 2$

17. $\int_1^\infty xe^{-2x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-2x}dx = \lim_{b\to\infty}\left(-\frac{x}{2}-\frac{1}{4}\right)e^{-2x}\Big|_1^b$
$= \lim_{b\to\infty}\left[\left(-\frac{b}{2}-\frac{1}{4}\right)e^{-2b}-\left(-\frac{3}{4}e^{-2}\right)\right]=\frac{3}{4}e^{-2}$

18. $\int_{-\infty}^0 x^2e^x\,dx = \lim_{b\to-\infty}\int_b^0 x^2e^x dx$
$= \lim_{b\to-\infty}\left((x^2-2x+2)e^{-2x}\right)\Big|_b^0$
$= \lim_{b\to-\infty}\left(\frac{b^2-2b+2}{e^{2b}}-2\right)=2$

19. $\int_1^\infty x\ln x\,dx = \lim_{b\to\infty}\int_1^b x\ln x\,dx = \lim_{b\to\infty}\left(\frac{x^2}{2}\ln 2-\frac{x^2}{4}\right)\Big|_1^b = \infty$

diverges

20. $\int_0^\infty (x+1)e^{-x}\,dx = \lim_{b\to\infty}\int_0^b (x+1)e^{-x}\,dx$
$= \lim_{b\to\infty}\left((-x-2)e^{-x}\right)\Big|_0^b$
$= \lim_{b\to\infty}\left(\frac{-b-2}{e^6}-(2)\right)$

21. $\int_{-\infty}^\infty e^{-|x|}\,dx = \lim_{b\to-\infty}\int_b^0 e^x dx+\lim_{b\to\infty}\int_0^b e^{-x}dx$
$= \lim_{b\to-\infty}(e^x)\Big|_b^0+\lim_{b\to\infty}(-e^{-x})\Big|_0^b$
$=1+[-(-1)]=2$

22. $\int_{-\infty}^\infty 2xe^{-x^2}\,dx = \lim_{b\to-\infty}\int_b^0 2xe^{-x^2}\,dx+\lim_{b\to\infty}\int_0^b 2xe^{-x^2}dx$
$= \lim_{b\to-\infty}\left(-e^{-x^2}\right)\Big|_b^0+\lim_{b\to\infty}\left(-e^{-x^2}\right)\Big|_0^b$
$=1+(-1)=0.$

23. $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}} = \lim_{b\to-\infty}\int_b^0\frac{dx}{e^x+e^{-x}}+\lim_{b\to\infty}\int_0^b\frac{dx}{e^x+e^{-x}}$
$\lim_{b\to-\infty}\left(\tan^{-1}(e^x)\right)\Big|_b^0+\lim_{b\to\infty}\left(\tan^{-1}(e^x)\right)\Big|_0^b$
$=[\tan^{-1}1-\tan^{-1}0]+\left[\lim_{b\to\infty}\tan^{-1}e^b-\tan^{-1}1\right]$
$=\left(\frac{\pi}{4}-0\right)+\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{2}$

24. $\int_{-\infty}^\infty e^{2x}\,dx = \lim_{b\to-\infty}\int_b^0 e^{2x}\,dx+\lim_{b\to\infty}\int_0^b e^{2x}\,dx$
$= \lim_{b\to-\infty}\left(\frac{e^{2x}}{2}\right)\Big|_b^0+\lim_{b\to\infty}\left(\frac{e^{2x}}{2}\right)\Big|_0^b=\infty$

diverges

25. **(a)** The integral has an infinite discontinuity at the interior point $x=1$.

(b) $\int_0^2\frac{dx}{1-x^2}=\lim_{b\to1^-}\int_0^b\frac{dx}{1-x^2}+\lim_{b\to1^+}\int_b^2\frac{dx}{1-x^2}$
$= \lim_{b\to1^-}\left(\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right)\Big|_0^b+\lim_{b\to1^+}\left(\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right)\Big|_b^2=\infty$

diverges

26. **(a)** The integral has an infinite discontinuity at the endpoint $x=1$.

(b) $\int_0^1\frac{dx}{\sqrt{1-x^2}}=\lim_{b\to1^-}\int_0^b\frac{dx}{\sqrt{1-x^2}}=\lim_{b\to1^-}\left(\sin^{-1}(x)\right)\Big|_0^b=\frac{\pi}{2}$

27. **(a)** The integral has an infinite discontinuity at the endpoint $x=0$.

(b) $\int_0^1\frac{x+1}{\sqrt{x^2+2x}}dx=\lim_{b\to0^+}\int_b^1\frac{x+1}{\sqrt{x^2+2x}}dx$
$=\lim_{b\to0^+}\left(\sqrt{x^2+2x}\right)\Big|_b^1=\sqrt{3}$

28. **(a)** The integral has an infinite discontinuity at the endpoint $x=0$.

(b) $\int_0^4\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx=\lim_{b\to0^+}\int_b^4\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$
$=\lim_{b\to0^+}\left(-2e^{-\sqrt{x}}\right)\Big|_b^4$
$=2-2e^{-2}$

29. **(a)** The integral has an infinite discontinuity at the endpoint $x=0$.

(b) $\int_0^1 x\ln(x)dx=\lim_{b\to0^+}\int_b^1 x\ln(x)dx$
$=\lim_{b\to0^+}\left(\frac{x^2}{2}\ln x-\frac{x^2}{4}\right)\Big|_b^1$
$=-1/4$

30. **(a)** The integral has an infinite discontinuity at the interior point $x=0$.

(b) $\int_{-1}^4\frac{dx}{\sqrt{|x|}}=\lim_{b\to0^-}\int_{-1}^b\frac{dx}{\sqrt{-x}}+\lim_{b\to0^+}\int_b^4\frac{dx}{\sqrt{x}}$
$=\lim_{b\to0^-}\left(-2\sqrt{-x}\right)\Big|_{-1}^b+\lim_{b\to0^+}\left(2\sqrt{x}\right)\Big|_b^4$
$=(0-(-2))+(4-0)=6$

31. $0\le\frac{1}{1+e^x}\le\frac{1}{e^x}$ on $[1,\infty)$, converges because $\int_1^\infty\frac{1}{e^x}dx$ converges.

32. $0\le\frac{1}{x^3+1}\le\frac{1}{x^3}$ on $[1,\infty)$, converges because $\int_1^\infty\frac{1}{x^3}dx$ converges.

33. $0\le\frac{1}{x}\le\frac{2+\cos x}{x}$ on $[\pi,\infty)$, diverges because $\int_\pi^\infty\frac{1}{x}dx$ diverges.

34. $\int_{-\infty}^\infty\frac{dx}{\sqrt{x^4+1}}=2\int_0^\infty\frac{dx}{\sqrt{x^4+1}}=2\int_0^1\frac{dx}{\sqrt{x^4+1}}+2\int_1^\infty\frac{dx}{\sqrt{x^4+1}}$
First integral is a proper integral. Second integral converges because $0\le\frac{1}{\sqrt{x^4+1}}\le\frac{1}{x^2}$ on $[1,\infty)$ and $\int_1^\infty\frac{1}{x^2}dx$ converges.

35. $\int_0^{\ln 2} y^{-2} e^{1/y}\, dy = \lim_{b\to 0^+} \int_b^{\ln 2} y^{-2} e^{1/y}\, dy = \lim_{b\to 0^+} \left(-e^{1/y}\right)\Big|_b^{\ln 2} = \infty$

diverges

36. $\int_0^4 \frac{dr}{\sqrt{4-r}} = \lim_{b\to 4^-} \int_0^b \frac{dr}{\sqrt{4-r}}$

$= \lim_{b\to 4^-} \left(-2\sqrt{4-r}\right)\Big|_0^b = 4$

37. $\int_0^\infty \frac{ds}{(1+s)\sqrt{s}} = \lim_{b\to 0^-} \int_b^1 \frac{ds}{(1+s)\sqrt{s}} + \lim_{b\to\infty} \int_1^b \frac{ds}{(1+s)\sqrt{s}}$

$= \lim_{b\to 0^-} \left(2\tan^{-1}\sqrt{s}\right)\Big|_b^1 + \lim_{b\to\infty}\left(2\tan^{-1}\sqrt{s}\right)\Big|_1^b$

$= 2\tan^{-1} 1 - \lim_{b\to 0^-}\left(2\tan^{-1} b\right)$

$+ \lim_{b\to\infty}\left(2\tan^{-1}\sqrt{b}\right) - 2\tan^{-1} 1$

$= \frac{\pi}{2} - 0 + \frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}$

38. $\int_1^2 \frac{du}{u\sqrt{u^2-1}} = \lim_{b\to 1^+} \int_b^2 \frac{du}{u\sqrt{u^2-1}}$

$= \lim_{b\to 1^+}\left(\tan^{-1}\sqrt{u^2-1}\right)\Big|_b^2 = \frac{\pi}{3}$

39. $\int_0^\infty \frac{16\tan^{-1} v}{1+v^2} dv = \lim_{b\to\infty} \int_0^b \frac{16\tan^{-1} v}{1+v^2} dv$

$= \lim_{b\to\infty}\left(8(\tan^{-1} v)^2\right)\Big|_0^b$

$= 2\pi^2$

40. $\int_{-\infty}^0 \theta e^\theta\, d\theta = \lim_{b\to-\infty} \int_b^0 \theta e^\theta\, d\theta$

$= \lim_{b\to-\infty}\left((\theta-1)e^\theta\right)\Big|_0^b = -1$

41. $\int_0^2 \frac{dt}{1-t} = \lim_{b\to 1^-} \int_0^b \frac{dt}{1-t} + \lim_{b\to 1^+} \int_b^2 \frac{dt}{1-t}$

$= \lim_{b\to 1^-}\left(-\ln|1-t|\right)\Big|_0^b + \lim_{b\to 1^+}\left(-\ln|1-t|\right)\Big|_b^2$

$= \infty + \infty$

diverges

42. $\int_{-1}^1 \ln(|w|)\, dw = 2\lim_{b\to 0^+} \int_b^1 \ln(w)\, dw = 2\lim_{b\to 0^+} [w\ln w - w]_b^1$

$= 2\lim_{b\to 0^+} [-1 + b(1-\ln b)] = \infty$

43. For $x \ge 0, y \ge 0$ on $[1, \infty)$.

Area $= \int_1^\infty \frac{\ln x}{x^2} dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2} dx$

Integrate $\int \frac{\ln x}{x^2} dx$ by parts.

$u = \ln x \qquad dv = \frac{dx}{x^2}$

$du = \frac{1}{x} dx \qquad v = -\frac{1}{x}$

$\int \frac{\ln x}{x^2} = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C$

Area $= \lim_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] = 1$

$\left(\text{Note that } \lim_{b\to\infty} \frac{\ln b}{b} = \lim_{b\to\infty} \frac{1/b}{1} = 0.\right)$

44. For $x \ge 0$, $y \ge 0$ on $[1, \infty)$.

Area $= \int_1^\infty \frac{\ln x}{x} dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x} dx$

Integrate $\int \frac{\ln x}{x} dx$ by letting $u = \ln x$, so $du = \frac{dx}{x}$.

$\int \frac{\ln x}{x} dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

Area $= \lim_{b\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_1^b = \lim_{b\to\infty} \frac{1}{2}(\ln b)^2 = \infty$

45. (a) Since f is even, $f(-x) = f(x)$. Let $u = -x$, $du = -dx$.

$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$

$= \int_\infty^0 f(-u)(-1)\,du + \int_0^\infty f(x)\,dx$

$= \int_0^\infty f(u)\,du + \int_0^\infty f(x)\,dx$

$= 2\int_0^\infty f(x)\,dx$

(b) Since f is odd, $f(-x) = -f(x)$. Let $u = -x$, then $du = -dx$.

$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$

$= \int_\infty^0 f(-u)(-1)\,du + \int_0^\infty f(x)\,dx$

$= \int_\infty^0 -f(u)(-1)\,du + \int_0^\infty f(x)\,dx$

$= \int_\infty^0 f(u)\,du + \int_0^\infty f(x)\,dx$

$= -\int_0^\infty f(u)\,du + \int_0^\infty f(x)\,dx$

$= -L + L = 0$

46. (a) $\int_0^\infty \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{2x\,dx}{x^2+1}$

$= \lim_{b\to\infty}\left[\ln(x^2+1)\right]_0^b$

$= \lim_{b\to\infty} \ln(b^2+1) = \infty$

Thus the integral diverges.

(b) Both $\int_0^\infty \frac{2x\,dx}{x^2+1}$ and $\int_{-\infty}^0 \frac{2x\,dx}{x^2+1}$ must converge in order for $\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$ to converge.

46. Continued

(c) $\lim_{b\to\infty}\int_{-b}^{b}\frac{2x\,dx}{x^2+1}=\lim_{b\to\infty}\left[\ln(x^2+1)\right]_{-b}^{b}$

$=\lim_{b\to\infty}\left[\ln(b^2+1)-\ln(b^2+1)\right]$

$=\lim_{b\to\infty}0=0.$

Note that $\frac{2x}{x^2+1}$ is an odd function so $\int_{-b}^{b}\frac{2x\,dx}{x^2+1}=0.$

(d) Although the limit in part (c) is correct, this is not a valid way to evaluate the indefinite intergral. In order for the integral to converge, there must be finite areas in both directions (toward ∞ and toward $-\infty$). In this case, there are infinite areas in both directions.

47. By symmetry, find the perimeter of one side, say for

$0\le x\le 1, y\ge 0.$

$y^{2/3}=1-x^{2/3}$

$y=(1-x^{2/3})^{3/2}$

$\frac{dy}{dx}=\frac{3}{2}(1-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)=-x^{-1/3}(1-x^{2/3})^{1/2}$

$\left(\frac{dy}{dx}\right)^2=x^{-2/3}(1-x^{2/3})=(x^{-2/3}-1)$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{x^{-2/3}}=x^{-1/3}$

$\int_0^1 x^{-1/3}\,dx=\lim_{b\to 0^+}\int_b^1 x^{-1/3}\,dx$

$=\lim_{b\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_b^1$

$=\lim_{b\to 0^+}\left[\frac{3}{2}-\frac{3}{2}b^{2/3}\right]=\frac{3}{2}$

Thus, the perimeter is $4\left(\frac{3}{2}\right)=6.$

48. False. See Theorem 6.

49. True. See Theorem 6.

50. C. $\int_1^{\infty}\frac{dx}{x^{1.01}}=\lim_{b\to\infty}\int_1^b\frac{dx}{x^{1.01}}=\lim_{b\to\infty}\left(-\frac{100}{x^{0.01}}\right)\Big|_1^b=100$

51. B. $\int_0^1\frac{dx}{x^{0.5}}=\lim_{b\to 0^+}\int_b^1\frac{dx}{x^{0.5}}=\lim_{b\to 0^+}\left(2\sqrt{x}\right)\Big|_b^1=2$

52. E. $\int_0^1\frac{dx}{x-1}=\lim_{b\to 1^-}\int_0^b\frac{dx}{x-1}=\lim_{b\to 1^-}\left(\ln(|x-1|)\right)\Big|_0^b=-\infty$

53. C. $\int_0^{\infty}\frac{dx}{x^2+1}=\lim_{b\to\infty}\int_0^b\frac{dx}{x^2+1}=\lim_{b\to\infty}\left(\tan^{-1}x\right)\Big|_0^b=\frac{\pi}{2}$

54. (a) $\int_1^{\infty}\frac{dx}{x^{0.5}}=\lim_{b\to\infty}\int_1^b\frac{dx}{x^{0.5}}=\lim_{b\to\infty}\left(2\sqrt{x}\right)\Big|_1^b=\infty$, or it diverges.

(b) $\int_1^{\infty}\frac{dx}{x}=\lim_{b\to\infty}\int_1^b\frac{dx}{x}=\lim_{b\to\infty}\left(\ln|x|\right)\Big|_1^b=\infty$, or it diverges.

(c) $\int_1^{\infty}\frac{dx}{x^{1.5}}=\lim_{b\to\infty}\int_1^b\frac{dx}{x^{1.5}}=\lim_{b\to\infty}\left(\frac{-2}{\sqrt{x}}\right)\Big|_1^b=2$

(d) $\int_1^{\infty}\frac{dx}{x^p}=\lim_{b\to\infty}\int_1^b\frac{dx}{x^p}$

$=\lim_{b\to\infty}\left(\frac{1}{1-p}\frac{1}{x^{1-p}}\right)\Big|_1^b$

$=\lim_{b\to\infty}\left(\frac{1}{1-p}\frac{1}{b^{1-p}}-\frac{1}{1-p}\frac{1}{1^{1-p}}\right)$

$=\lim_{b\to\infty}\left(\frac{1}{1-p}\left(\frac{1}{b^{1-p}}-1\right)\right)$

(e) $\int_1^{\infty}\frac{dx}{x^p}$, $p>1$ (so $p-1>0$)

$=\lim_{b\to\infty}\left(\frac{1}{1-p}\bullet\frac{1}{b^{p-1}}-\frac{1}{1-p}\right)$

$=\frac{1}{1-p}\bullet 0-\frac{1}{1-p}$

$=\frac{1}{p-1}$

$\int_1^{\infty}\frac{dx}{x^p}$, $p<1$ (so $p-1<0$)

$=\lim_{b\to\infty}\left(\frac{1}{1-p}\bullet\frac{1}{b^{p-1}}-\frac{1}{1-p}\right)$

$=\lim_{b\to\infty}\left(\frac{1}{1-p}\bullet b^{1-p}-\frac{1}{1-p}\right)$

$=\infty$

55. (a) $A(x)=\frac{\pi}{4}e^{2x}$

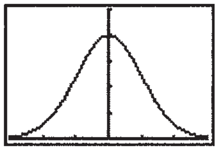
(b) $V=\int_{-\infty}^{\ln 2}A(x)\,dx=\int_{-\infty}^{\ln 2}\frac{\pi}{4}e^{2x}\,dx$

(c) $V=\int_{-\infty}^{\ln 2}\frac{\pi}{4}e^{2x}\,dx=\lim_{b\to-\infty}\int_b^{\ln 2}\frac{\pi}{4}e^{2x}\,dx$

$=\lim_{b\to-\infty}\left(\frac{\pi e^{2x}}{8}\right)\Big|_b^{\ln 2}$

$=\frac{\pi 4}{8}-0=\frac{\pi}{2}$

56. **(a)** $f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$



[–3, 3] by [0, 0.5]

f is increasing on $(-\infty, 0]$, f is decreasing on $[0, \infty)$.

f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$

(b) $\text{NINT}\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, x, -1, 1\right)\approx 0.683$

$\text{NINT}\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, x, -2, 2\right)\approx 0.954$

$\text{NINT}\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, x, -3, 3\right)\approx 0.997$

(c) Part (b) suggests that as b increases, the integral approaches 1. We can make $\int_{-b}^{b} f(x)dx$ as close to 1 as we want by choosing $b > 1$ large enough. Also we can make $\int_{-b}^{\infty} f(x)dx$ and $\int_{\infty}^{-b} f(x)dx$ as small as we want by choosing b large enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$). Thus, for $b > 1$,

$$\int_b^{\infty} f(x)dx < \int_b^{\infty} e^{-x/2}\,dx.$$

$$\int_b^{\infty} e^{-x/2}\,dx = \lim_{c\to\infty}\int_b^{c} e^{-x/2}\,dx$$

$$= \lim_{c\to\infty}\left[-2e^{-x/2}\right]_b^c$$

$$= \lim_{c\to\infty}[-2e^{-c/2}+2e^{-b/2}]$$

$$= 2e^{-b/2}$$

As $b \to \infty$, $2e^{-b/2} \to 0$, so for large enough b, $\int_b^{\infty} f(x)dx$ is as small as we want. Likewise, for large enough b, $\int_{-\infty}^{-b} f(x)dx$ is as small as we want.

57. **(a)** For $x \ge 6, x^2 \ge 6x$, so $e^{-x^2} \le e^{-6x}$

$$\int_6^{\infty} e^{-x^2}dx \le \int_6^{\infty} e^{-6x}\,dx$$

$$= \lim_{b\to\infty}\int_6^{b} e^{-6x}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{6}e^{-6x}\right]_6^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{6}e^{-6b}+\frac{1}{6}e^{-36}\right)$$

$$= \frac{1}{6}e^{-36} < 4\times 10^{-17}$$

(b) $$\int_1^{\infty} e^{-x^2}\,dx = \int_1^{6} e^{-x^2}\,dx + \int_6^{\infty} e^{-x^2}\,dx$$

$$\le \int_1^{6} e^{-x^2}\,dx + 4\times 10^{-17}$$

Thus, from part (a) we have shown that the error is bounded by 4×10^{-17}.

(c) $\int_1^{\infty} e^{-x^2}\,dx \approx \text{NINT}(e^{-x^2}, x, 1.6) \approx 0.1394027926$

(This agrees with Figure 8.16.)

(d) $$\int_0^{\infty} e^{-x^2}\,dx = \int_0^{3} e^{x^2}\,dx + \int_3^{\infty} e^{x^2}\,dx$$

$$\le \int_0^{3} e^{-x^2}\,dx + \int_3^{\infty} e^{-3x}\,dx$$

since $x^2 \ge 3x$ for $x > 3$.

$$\int_3^{\infty} e^{-3x}\,dx = \lim_{b\to\infty}\int_3^{b} e^{-3x}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_3^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3b}+\frac{1}{3}e^{-9}\right)$$

$$= \frac{1}{3}e^{-9} \approx 0.000041 < 0.000042$$

58. Suppose $0 \le f(x) \le g(x)$ for all $x \ge a$.
From the properties of integrals, for any $b > a$,
$\int_a^b f(x) \le \int_a^b g(x)dx.$

If the infinite integral of g converges, then taking the limit in the above inequality as $b \to \infty$ shows that the infinite integral of f is bounded above by the infinite integral integral of g. Therefore, the infinite integral of f must be finite and it converges. If the infinite integral of f diverges, it must grow to infinity. So taking the limit in the above inequality as $b \to \infty$ shows that the infinite integral of g must also diverage to infinity.

59. **(a)** For $n = 0$:

$$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\int_0^{b} e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$$

$$= \lim_{b\to\infty}[-e^{-b}+1] = 1$$

59. Continued

For $n = 1$:

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(\left[-xe^{-x}\right]_0^b + \int_0^b e^{-x}\,dx\right)$$
$$= \lim_{b\to\infty}\left(-\frac{b}{e^b}\right) + \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{1}{e^b}\right) + 1 = 1$$

For $n = 2$;

$u = x^2 \qquad dv = e^{-x}\,dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(\left[-x^2e^{-x}\right]_0^b + \int_0^b 2xe^{-x}\,dx\right)$$
$$= \lim_{b\to\infty}\left(-\frac{b^2}{e^b}\right) + 2\lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{2b}{e^b}\right) + 2(1)$$
$$= \lim_{b\to\infty}\left(-\frac{2b}{e^b}\right) + 2 = 2$$

(b) Evaluate $\int x^n e^{-x}\,dx$ using integration by parts

$u = x^n \qquad dv = e^{-x}\,dx$

$du = nx^{n-1} \qquad v = -e^{-x}$

$$\int x^n e^{-x}\,dx = -x^n e^{-x} + \int nx^{n-1}e^{-x}\,dx$$

$$f(n+1) = \int_0^\infty x^n e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left[-x^ne^{-x}\right]_0^b + \int_0^\infty nx^{n-1}e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{b^n}{e^b}\right) + n\int_0^\infty x^{n-1}e^{-x}\,dx$$
$$= nf(n)$$

(Note: apply L'Hôpital's Rule n times to show that $\lim_{b\to\infty}\left(-\frac{b^n}{e^b}\right) = 0.$)

(c) Since $f(n+1) = nf(n)$,

$f(n+1) = n(n-1)\cdots f(1) = n!$; thus

$\int_0^\infty x^n e^{-x}\,dx$ converges for all integers $n \ge 0$.

60. (a) On a grapher, plot $\text{NINT}\left(\frac{\sin x}{x}, x, 0, x\right)$ or create a table of values. For large values of $x, f(x)$ appears to approach approximately 1.57.

(b) The integral is an infinite sum of finite areas that *alternate in sign, decrease in absolute value*, and tend to *a limit of zero*. The partial sums must therefore close in on some limit. (Figure 9.18 in Section 9.5 shows a diagram of what this convergence looks like.)

61. (a) $$\int_{-\infty}^1 \frac{dx}{1+x^2} = \lim_{b\to-\infty}\int_b^1 \frac{dx}{1+x^2}$$
$$= \lim_{b\to-\infty}\left[\tan^{-1}x\right]_b^1$$
$$= \lim_{b\to-\infty}(\tan^{-1}1 - \tan^{-1}b)$$
$$= \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{x\to\infty}\int_1^b \frac{dx}{1+x^2}$$
$$= \lim_{b\to\infty}\left[\tan^{-1}x\right]_1^b$$
$$= \lim_{b\to\infty}[\tan^{-1}b - \tan^{-1}1]$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{3\pi}{4} + \frac{\pi}{4} = \pi$$

(b) $$\int_{-\infty}^c f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^c f(x)dx$$
$$\int_c^\infty f(x)dx = \int_c^0 f(x)dx + \int_0^\infty f(x)dx$$

Thus,

$$\int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$
$$= \int_{-\infty}^0 f(x)dx + \int_0^c f(x)dx + \int_c^0 f(x)dx + \int_0^\infty f(x)dx$$
$$= \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx, \text{ because}$$
$$\int_0^c f(x)dx + \int_c^0 f(x)dx = \int_0^c f(x)dx - \int_0^c f(x)dx = 0.$$

Quick Quiz Sections 8.3 and 8.4

1. E. $\lim_{x\to\infty}\frac{0.1x^3}{x^2} = \lim_{x\to\infty}\frac{0.3x^2}{2x} = \lim_{x\to\infty}\frac{0.6x}{2} = \infty.$

2. C. Since $\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$ for $p > 1$, and $p + 1 > 1$ when $p > 0$.

3. B. $\int_0^1 \frac{dx}{x^{p+1}} = C$ when $p < 0$.