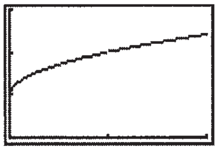
4.

x	$\left(1+\dfrac{1}{x}\right)^x$
−1.1	13.981
−1.01	105.77
−1.001	1007.9
−1.0001	10010

As $x \to -1^-$, $\left(1+\dfrac{1}{x}\right)^x$ goes to ∞.

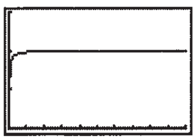
5.



[0, 2] by [0, 3]

As $t \to 1$, $\dfrac{t-1}{\sqrt{t}-1}$ approaches 2.

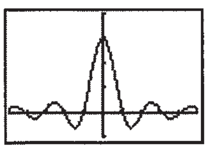
6.



[0, 500] by [0, 3]

As $x \to \infty$, $\dfrac{\sqrt{4x^2+1}}{x+1}$ approaches 2.

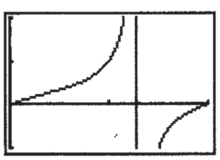
7.



[−5, 5] by [−1, 4]

As $x \to 0$, $\dfrac{\sin 3x}{x}$ approaches 3.

8.



[0, π] by [−1, 2]

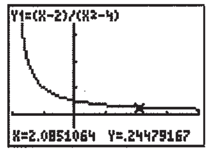
As $\theta \to \dfrac{\pi}{2}$, $\dfrac{\tan\theta}{2+\tan\theta}$ approaches 1.

9. $y = \dfrac{1}{h}\sin h = \dfrac{\sin h}{h}$

10. $y = (1+h)^{1/h}$

Section 8.2 Exercises

1. $\lim\limits_{x\to 2}\left(\dfrac{x-2}{x^2-4}\right)$ appears to be about $\dfrac{1}{4}$;

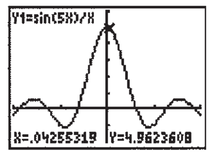


[−2, 4] by [−1, 4]

By L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right) = \lim_{x\to 2}\frac{1}{2x} = \frac{1}{4}$$

2. $\lim\limits_{x\to 0}\left(\dfrac{\sin(5x)}{x}\right)$ appears to be about 5;

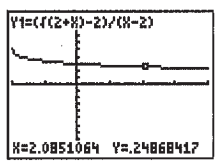


[−2, 2] by [−2, 6]

By L'Hôpital's Rule:

$$\lim_{x\to 0}\left(\frac{\sin(5x)}{x}\right) = \lim_{x\to 0}\frac{5\cos(5x)}{1} = 5\cos(0) = 5$$

3. $\lim\limits_{x\to 2}\left(\dfrac{\sqrt{2+x}-2}{x-2}\right)$ appears to be about $\dfrac{1}{4}$;



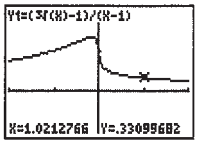
[−2, 4] by [−1, 1]

By L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\right) = \lim_{x\to 2}\left(\frac{\frac{1}{2}(2+x)^{-1/2}}{1}\right) = \frac{1}{2}(2+2)^{-1/2}$$

$$= \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

4. $\lim_{x\to 1}\left(\dfrac{\sqrt[3]{x}-1}{x-1}\right)$ appears to be about $\dfrac{1}{3}$;



[−2, 2] by [−1, 2]

By L'Hôpital's Rule:

$$\lim_{x\to 1}\left(\frac{\sqrt[3]{x}-1}{x-1}\right)=\lim_{x\to 1}\left(\frac{\frac{1}{3}(x)^{-2/3}}{1}\right)=\frac{1}{3}(1)^{-2/3}=\frac{1}{3}$$

5. $$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)=\lim_{x\to 0}\left(\frac{\sin x}{2x}\right)=\lim_{x\to 0}\left(\frac{\cos x}{2}\right)$$
$$=\frac{\cos(0)}{2}=\frac{1}{2}$$

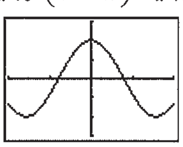
6. $$\lim_{\theta\to\pi/2}\left(\frac{1-\sin\theta}{1+\cos(2\theta)}\right)=\lim_{\theta\to\pi/2}\left(\frac{-\cos\theta}{-2\sin(2\theta)}\right)$$
$$=\lim_{\theta\to\pi/2}\left(\frac{\sin\theta}{-4\cos(2\theta)}\right)=\frac{\sin\frac{\pi}{2}}{-4\cos\pi}=\frac{1}{4}$$

7. $$\lim_{t\to 0}\left(\frac{\cos t-1}{e^t-t-1}\right)=\lim_{t\to 0}\left(\frac{-\sin t}{e^t-1}\right)$$
$$=\lim_{t\to 0}\left(\frac{-\cos t}{e^t}\right)=\frac{-\cos(0)}{e^0}=-1$$

8. $$\lim_{x\to 2}\left(\frac{x^2-4x+4}{x^3-12x+16}\right)=\lim_{x\to 2}\left(\frac{2x-4}{3x^2-12}\right)=\lim_{x\to 2}\left(\frac{2}{6x}\right)$$
$$=\frac{2}{6(2)}=\frac{1}{6}$$

9. (a) $$\lim_{x\to 0^-}\left(\frac{\sin 4x}{\sin 2x}\right)=\lim_{x\to 0^-}\left(\frac{4\cos(4x)}{2\cos(2x)}\right)=\frac{4\cos(0)}{2\cos(0)}=2$$

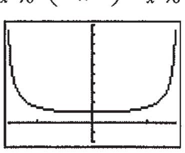
(b) $$\lim_{x\to 0^+}\left(\frac{\sin 4x}{\sin 2x}\right)=\lim_{x\to 0^+}\left(\frac{4\cos(4x)}{2\cos(2x)}\right)=\frac{4\cos(0)}{2\cos(0)}=2$$



[−2, 2] by [−3, 3]

10. (a) $$\lim_{x\to 0^-}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2 x}{1}\right)=1$$

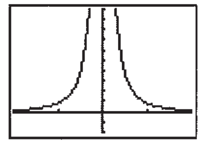
(b) $$\lim_{x\to 0^+}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2 x}{1}\right)=1$$



[−1.5, 1.5] by [−2, 10]

11. (a) $$\lim_{x\to 0^-}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^-}\left(\frac{\cos x}{3x^2}\right)=\infty$$

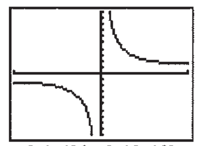
(b) $$\lim_{x\to 0^+}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^+}\left(\frac{\cos x}{3x^2}\right)=\infty$$



[−2, 2] by [−2, 10]

12. (a) $$\lim_{x\to 0^-}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2 x}{2x}\right)=-\infty$$

(b) $$\lim_{x\to 0^+}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2 x}{2x}\right)=\infty$$



[−1, 1] by [−10, 10]

13. Left:

$$\lim_{x\to\pi^-}\left(\frac{\csc x}{1+\cot x}\right)=\frac{\infty}{-\infty}$$

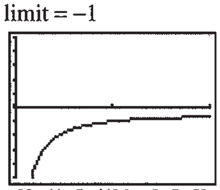
$$\lim_{x\to\pi^-}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=\lim_{x\to\pi^-}\left(\frac{\cot x}{\csc x}\right)$$
$$=\lim_{x\to\pi^-}\frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)}$$
$$=\lim_{x\to\pi^-}\cos x=-1$$

Right:

$$\lim_{x\to\pi^+}\left(\frac{\csc x}{1+\cot x}\right)=\frac{-\infty}{\infty}$$

$$\lim_{x\to\pi^+}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=\lim_{x\to\pi^+}\left(\frac{\cot x}{\csc x}\right)$$
$$=\lim_{x\to\pi^+}\frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)}$$
$$=\lim_{x\to\pi^+}\cos x=-1$$

Y1 = (1/sin(x))/(1 + cos(x)/sin(x))
limit = −1



[$3\pi/4$, $5\pi/4$] by [−5, 5]

14. Left:

$$\lim_{x\to\pi/2^-}\left(\frac{1+\sec x}{\tan x}\right)=\frac{\infty}{\infty}$$

$$\lim_{x\to\pi/2^-}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=\lim_{x\to\pi/2^-}\left(\frac{\tan x}{\sec x}\right)=\lim_{x\to\pi/2^-}\frac{\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)}=\lim_{x\to\pi/2^-}\sin x=1$$

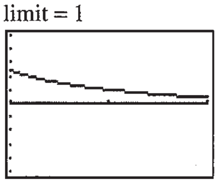
Right:

$$\lim_{x\to\pi/2^+}\left(\frac{1+\sec x}{\tan x}\right)=\frac{-\infty}{-\infty}$$

$$\lim_{x\to\pi/2^+}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=\lim_{x\to\pi/2^+}\left(\frac{\tan x}{\sec x}\right)=\lim_{x\to\pi/2^+}\frac{\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)}=\lim_{x\to\pi/2^+}\sin x=1$$

$Y1 = (1 + 1/\cos(x))/\tan(x)$

limit = 1



$[\pi/4, 3\pi/4]$ by $[-5, 5]$

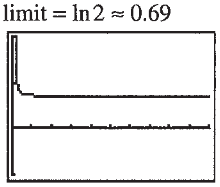
15. $\displaystyle\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}=\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\left(\frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}}\right)=\lim_{x\to\infty}\left(\frac{x\ln 2}{x+1}\right)=\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\left(\frac{\ln 2}{1}\right)=\ln 2$$

$Y1 = \ln(x + 1)/(\ln(x)/\ln(2))$

limit $= \ln 2 \approx 0.69$



$[0, 100]$ by $[-1, 2]$

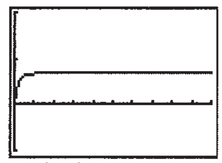
16. $\displaystyle\lim_{x\to\infty}\left(\frac{5x^2-3x}{7x^2+1}\right)=\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\left(\frac{10x-3}{14x}\right)=\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\left(\frac{10}{14}\right)=\frac{5}{7}$$

$Y1 = (5x^2 - 3x)/(7x^2 + 1)$

limit $=\frac{5}{7}\approx 0.71$



$[0, 100]$ by $[-1, 2]$

17. $\displaystyle\lim_{x\to 0^+}(x\ln x)=0\cdot\infty$

$$\lim_{x\to 0^+}\left(\frac{\ln x}{1/x}\right)=\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)=\lim_{x\to 0^+}(-x)=0$$

18. $\displaystyle\lim_{x\to\infty}\left(x\tan\left(\frac{1}{x}\right)\right)=\infty\cdot 0$

Let $h=\frac{1}{x}$. Then $h\to 0$ as $x\to\infty$, and

$$\lim_{x\to\infty}x\tan\left(\frac{1}{x}\right)=\lim_{h\to 0}\left(\frac{1}{h}\tan(h)\right)=\lim_{h\to 0}\left(\frac{\tan(h)}{h}\right)=\frac{\infty}{\infty}$$

$$\lim_{h\to 0}\frac{\sec^2 h}{1}=\sec^2(0)=1$$

19. $$\lim_{x\to 0^+}(\csc x-\cot x+\cos x)=\lim_{x\to 0^+}(\csc x-(\cot x-\cos x))=\infty-\infty$$

$$=\lim_{x\to 0^+}\frac{1-\cos x+\cos x\sin x}{\sin x}=\lim_{x\to 0^+}\frac{\sin x-\sin^2 x+\cos^2 x}{\cos x}=\frac{0-0+1}{1}=1.$$

20. $\displaystyle\lim_{x\to\infty}(\ln(2x)-\ln(x+1))=\infty-\infty$

$$\ln(2x)-\ln(x+1)=\ln\left(\frac{2x}{x+1}\right)$$

$$\lim_{x\to\infty}\left(\frac{2x}{x+1}\right)=\lim_{x\to\infty}\frac{2}{1}=2$$

Therefore:

$$\lim_{x\to\infty}(\ln(2x)-\ln(x+1))=\lim_{x\to\infty}\left(\ln\left(\frac{2x}{x+1}\right)\right)=\ln(2)$$

21. $\lim_{x\to 0}(e^x+x)^{1/x}=(1+0)^\infty=1^\infty$

$$\ln f(x)=\ln\left[(e^x+x)^{1/x}\right]=\frac{\ln(e^x+x)}{x}$$

$$\lim_{x\to 0}\left(\frac{\ln(e^x+x)}{x}\right)=\lim_{x\to 0}\frac{\frac{1}{e^x+x}\bullet(e^x+1)}{1}=\lim_{x\to 0}\frac{e^x+1}{e^x+x}=2$$

Therefore:

$$\lim_{x\to 0}(e^x+x)^{1/x}=\lim_{x\to 0}f(x)=\lim_{x\to 0}e^{\ln f(x)}=e^2$$

22. $\lim_{x\to 1}x^{1/(x-1)}=1^\infty$

$$\ln f(x)=\ln\left[x^{1/(x-1)}\right]=\frac{\ln x}{x-1}$$

$$\lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{x\to 1}\frac{1/x}{1}=1$$

Therefore:

$$\lim_{x\to 1}x^{1/(x-1)}=\lim_{x\to 1}f(x)=\lim_{x\to 1}e^{\ln f(x)}=e^1=e$$

23. $\lim_{x\to 1}(x^2-2x+1)^{x-1}=(1^2-2(1)+1)^{1-1}=0^0$

$$\ln f(x)=(x-1)\ln(x^2-2x+1)=\frac{\ln(x^2-2x+1)}{1/(x-1)}$$

$$\lim_{x\to 1}\frac{\ln(x^2-2x+1)}{1/(x-1)}=\lim_{x\to 1}\frac{\frac{2}{x-1}}{-1/(x-1)^2}=\lim_{x\to 1}\left(\frac{-2(x-1)^2}{x-1}\right)=0$$

Therefore:

$$\lim_{x\to 1}(x^2-2x+1)^{x-1}=\lim_{x\to 1}f(x)=\lim_{x\to 1}e^{\ln f(x)}=e^0=1$$

24. $\lim_{x\to 0^+}(\sin x)^x=0^0$

$$\ln f(x)=\ln[(\sin x)^x]=x\ln(\sin x)=\frac{\ln(\sin x)}{1/x}$$

$$\lim_{x\to 0^+}\frac{\ln(\sin x)}{1/x}=\lim_{x\to 0^+}\left(\frac{\frac{1}{\tan x}}{-1/x^2}\right)=\lim_{x\to 0^+}\left(-\frac{x^2}{\tan x}\right)$$

$$=\lim_{x\to 0^+}\left(-\frac{2x}{\sec^2 x}\right)=-\frac{0}{1}=0$$

Therefore:

$$\lim_{x\to 0^+}(\sin x)^x=\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}e^{\ln f(x)}=e^0=1$$

25. $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=(1+\infty)^0=\infty^0$

$$\ln f(x)=x\ln\left(1+\frac{1}{x}\right)=\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$$

$$\lim_{x\to 0^+}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}=\lim_{x\to 0^+}\left(\frac{\frac{-1}{x(x+1)}}{-1/x^2}\right)$$

$$=\lim_{x\to 0^+}\frac{x^2}{x(x+1)}$$

$$=\lim_{x\to 0^+}\frac{x}{x+1}=0.$$

Therefore:

$$\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}e^{\ln f(x)}=e^0=1$$

26. $\lim_{x\to\infty}(\ln x)^{1/x}=\infty^0$

$$\ln f(x)=\ln[(\ln x)^{1/x}]=\frac{\ln(\ln x)}{x}$$

$$\lim_{x\to\infty}\frac{\ln(\ln x)}{x}=\lim_{x\to\infty}\left(\frac{\frac{1}{\ln x}\bullet\frac{1}{x}}{1}\right)=\lim_{x\to\infty}\left(\frac{1}{x\ln x}\right)=0.$$

Therefore:

$$\lim_{x\to\infty}(\ln x)^{1/x}=\lim_{x\to\infty}f(x)=\lim_{x\to\infty}e^{\ln f(x)}=e^0=1$$

27. (a)

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.1513	0.2303	0.0345	0.00461	0.00058

Estimate the limit to be 0.

(b) $\lim_{x\to\infty}\frac{\ln x^5}{x}=\lim_{x\to\infty}\frac{5\ln x}{x}=\lim_{x\to\infty}\frac{5/x}{1}=\frac{0}{1}=0$

28. (a)

x	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$f(x)$	0.1585	0.1666	0.1667	0.1667	0.1667

Estimate the limit to be $\frac{1}{6}$.

(b) $\lim_{x\to 0^+}\frac{x-\sin x}{x^3}=\lim_{x\to 0^+}\frac{1-\cos x}{3x^2}$

$$=\lim_{x\to 0^+}\frac{\sin x}{6x}$$

$$=\lim_{x\to 0^+}\frac{\cos x}{6}$$

$$=\frac{1}{6}$$

29. Let $f(\theta)=\dfrac{\sin 3\theta}{\sin 4\theta}$.

θ	$\pm10^0$	$\pm10^{-1}$	$\pm10^{-2}$	$\pm10^{-3}$	$\pm10^{-4}$
$f(\theta)$	−0.1865	0.7589	0.7501	0.7500	0.7500

Estimate the limit to be $=\dfrac{3}{4}$.

$$\lim_{\theta\to0}\frac{\sin 3\theta}{\sin 4\theta}=\lim_{\theta\to0}\frac{3\cos 3\theta}{4\cos 4\theta}=\frac{3}{4}$$

30. Let $f(t)=\dfrac{1}{\sin t}-\dfrac{1}{t}=\dfrac{t-\sin t}{t\sin t}$.

t	$\pm10^0$	$\pm10^{-1}$	$\pm10^{-2}$	$\pm10^{-3}$
$f(t)$	$\pm$ 0.1884	$\pm$ 0.0167	$\pm$ 0.0017	$\pm$ 0.00017

Estimate the limit to be 0.

$$\lim_{t\to0}\left(\frac{1}{\sin t}-\frac{1}{t}\right)=\lim_{t\to0}\frac{t-\sin t}{t\sin t}$$
$$=\lim_{t\to0}\frac{1-\cos t}{t\cos t+\sin t}$$
$$=\lim_{t\to0}\frac{\sin t}{-t\sin t+\cos t+\cos t}=0$$

31. Let $f(x)=(1+x)^{1/x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.2710	1.0472	1.0069	1.0009	1.0001

Estimate the limit to be 1.

$$\ln f(x)=\frac{\ln(1+x)}{x}$$

$$\lim_{x\to\infty}\frac{\ln(1+x)}{x}=\lim_{x\to\infty}\frac{\frac{1}{1+x}}{1}=\frac{0}{1}=0$$
$$\lim_{x\to\infty}(1+x)^{1/x}=\lim_{x\to\infty}f(x)=\lim_{x\to\infty}e^{\ln f(x)}=e^0=1$$

32. Let $f(x)=\dfrac{x-2x^2}{3x^2+5x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	−0.5429	−0.6525	−0.6652	−0.6665	−0.6667

Estimate the limit to be $-\dfrac{2}{3}$.

$$\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}=\lim_{x\to\infty}\frac{1-4x}{6x+5}=\lim_{x\to\infty}-\frac{4}{6}=-\frac{2}{3}.$$

33. $\displaystyle\lim_{\theta\to0}\frac{\sin\theta^2}{\theta}=\lim_{\theta\to0}\frac{2\theta\cos\theta^2}{1}=(2)(0)\cos(0)^2=0$

34.
$$\lim_{t\to1}\frac{t-1}{\ln t-\sin\pi t}=\lim_{t\to1}\frac{1}{\frac{1}{t}-\pi\cos\pi t}$$
$$=\frac{1}{1-\pi(-1)}$$
$$=\frac{1}{\pi+1}$$

35.
$$\lim_{x\to\infty}\frac{\log_2 x}{\log_3(x+3)}=\lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{(x+3)\ln 3}}$$
$$=\lim_{x\to\infty}\frac{(x+3)\ln 3}{x\ln 2}$$
$$=\lim_{x\to\infty}\frac{x\ln 3+3\ln 3}{x\ln 2}$$
$$=\lim_{x\to\infty}\frac{\ln 3}{\ln 2}$$
$$=\frac{\ln 3}{\ln 2}$$

36.
$$\lim_{y\to0^+}\frac{\ln(y^2+2y)}{\ln y}=\lim_{y\to0^+}\frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}}$$
$$=\lim_{y\to0^+}\frac{y(2y+2)}{y^2+2y}$$
$$=\lim_{y\to0^+}\frac{(2y^2+2y)}{y^2+2y}$$
$$=\lim_{y\to0^+}\frac{4y+2}{2y+2}$$
$$=\frac{4(0)+2}{2(0)+2}=\frac{2}{2}=1$$

37.
$$\lim_{y\to\pi/2}\left(\frac{\pi}{2}-y\right)\tan y=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\sin y}{\cos y}$$
$$=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\cos y+(-1)\sin y}{-\sin y}$$
$$=\frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\cos\frac{\pi}{2}+(-1)\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}}$$
$$=\frac{(-1)(1)}{-(1)}=1$$

38. $\lim\limits_{x\to 0^+}(\ln x - \ln\sin x) = \lim\limits_{x\to 0^+}\ln\dfrac{x}{\sin x}$

Let $f(x) = \dfrac{x}{\sin x}$.

$\lim\limits_{x\to 0^+}\dfrac{x}{\sin x} = \lim\limits_{x\to 0^+}\dfrac{1}{\cos x} = 1$. Therefore,

$\lim\limits_{x\to 0^+}(\ln x - \ln\sin x) = \lim\limits_{x\to 0^+}\ln f(x) = \ln 1 = 0$

39. This problem does not require L'Hôpital's Rule.

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = \lim_{x\to 0^+}\frac{1-\sqrt{x}}{x} = \infty$$

40. The limit leads to the indeterminate form ∞^0.

Let $f(x) = \left(\dfrac{1}{x^2}\right)^x$.

$$\ln\left(\frac{1}{x^2}\right)^x = x\ln\left(\frac{1}{x^2}\right) = \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}$$

$$\lim_{x\to 0}\frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{-2/x^3}{1/x^2}}{-1/x^2} = \lim_{x\to 0}2x = 0$$

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right)^x = \lim_{x\to 0}e^{\ln f(x)} = e^0 = 1$$

41. $\lim\limits_{x\to\pm\infty}\dfrac{3x-5}{2x^2-x+2} = \lim\limits_{x\to\pm\infty}\dfrac{3}{4x-1} = 0$

42. $\lim\limits_{x\to 0}\dfrac{\sin 7x}{\tan 11x} = \lim\limits_{x\to 0}\dfrac{7\cos 7x}{11\sec^2 11x} = \dfrac{7}{11}$

43. The limit leads to the indeterminate form ∞^0.

Let $f(x) = (1+2x)^{1/(2\ln x)}$.

$$\ln(1+2x)^{1/(2\ln x)} = \frac{\ln(1+2x)}{2\ln x}$$

$$\lim_{x\to\infty}\frac{\ln(1+2x)}{2\ln x} = \lim_{x\to\infty}\frac{\frac{2}{1+2x}}{\frac{2}{x}} = \lim_{x\to\infty}\frac{x}{1+2x} = \lim_{x\to\infty}\frac{1}{2} = \frac{1}{2}$$

$$\lim_{x\to\infty}(1+2x)^{1/(2\ln x)} = \lim_{x\to\infty}e^{\ln f(x)} = e^{1/2} = \sqrt{e}$$

44. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\cos x)^{\cos x}$.

$$\ln(\cos x)^{\cos x} = (\cos x)\ln(\cos x) = \frac{\ln(\cos x)}{\sec x}$$

$$\lim_{x\to\pi/2^-}\frac{\ln(\cos x)}{\sec x} = \lim_{x\to\pi/2^-}\frac{\frac{-\sin x}{\cos x}}{\sec x\tan x} = \lim_{x\to\pi/2^-}\frac{-\tan x}{\sec x\tan x} = \lim_{x\to\pi/2^-}-\cos x = 0$$

$$\lim_{x\to\frac{\pi}{2}^-} = (\cos x)^{\cos x} = \lim_{x=\frac{\pi}{2}^-}e^{\ln f(x)} = e^0 = 1$$

45. The limit leads to the indeterminate form 1^∞.

Let $f(x) = (1+x)^{1/x}$.

$$\ln(1+x)^{1/x} = \frac{\ln(1+x)}{x}$$

$$\lim_{x\to 0^+}\frac{\ln(1+x)}{x} = \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1} = 1$$

$$\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+}e^{\ln f(x)} = e^1 = e$$

46. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\sin x)^{\tan x}$

$$\ln(\sin x)^{\tan x} = \tan x\ln(\sin x) = \frac{\ln(\sin x)}{\cot x}$$

$$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x} = \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to 0^+}(-\sin x\cos x) = 0$$

$$\lim_{x\to 0^+}(\sin x)^{\tan x} = \lim_{x\to 0^+}e^{\ln f(x)} = e^0 = 1$$

47. The limit leads to the indeterminate form $1^{-\infty}$.

Let $f(x) = x^{1/(1-x)}$.

$$\ln x^{1/(1-x)} = \frac{\ln x}{1-x}$$

$$\lim_{x\to 1^+}\frac{\ln x}{1-x} = \lim_{x\to 1^+}\frac{\frac{1}{x}}{-1} = -1$$

$$\lim_{x\to 1^+}x^{1/(1-x)} = \lim_{x\to 1^+}e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

48. $\int_x^{2x} \frac{dt}{t} = \left[\ln|t|\right]_x^{2x} = \ln|2x| - \ln|x| = \ln\left|\frac{2x}{x}\right|$

$$\lim_{x\to\infty} \int_x^{2x} \frac{dt}{t} = \lim_{x\to\infty} \ln\left|\frac{2x}{x}\right| = \lim_{x\to\infty} \ln 2 = \ln 2$$

49. $\lim_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x\to 1} \frac{3x^2}{12x^2-1}$
$= 3/11$

50. $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty} \frac{4x+3}{3x^2+1}$
$= \lim_{x\to\infty} \frac{4}{6x} = 0$

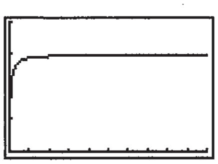
51. $\lim_{x\to 1} \frac{\int_1^x \cos t\, dt}{x^2-1} = \lim_{x\to 1} \frac{\sin x - \sin 1}{x^2-1}$
$= \lim_{x\to 1} \frac{\cos x}{2x} = \frac{\cos 1}{2}$

52. $\lim_{x\to 1} \frac{\int_1^x \frac{dt}{t}}{x^3-1} = \lim_{x\to 1} \frac{\ln x - \ln 1}{x^3-1}$
$= \lim_{x\to 1} \frac{1/x}{3x^2} = 1/3$

53. **(a)** L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$$\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{(9/2)(9x+1)^{-1/2}}{(1/2)(x+1)^{-1/2}}$$
$$= \lim_{x\to\infty} \frac{9\sqrt{x+1}}{\sqrt{9x+1}}$$

(b)



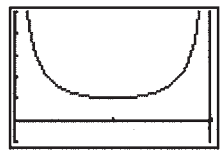
[0, 100] by [0, 4]
The limit appears to be 3.

(c) $\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{\sqrt{9+\frac{1}{x}}}{\sqrt{1+\frac{1}{x}}} = \frac{\sqrt{9}}{\sqrt{1}} = 3$

54. **(a)** L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$$\lim_{x\to\pi/2} \frac{\sec x}{\tan x} = \lim_{x\to\pi/2} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to\pi/2} \frac{\tan x}{\sec x}$$

(b) Y1 = (1/cos(x))/tan(x)



[0, π] by [–1, 5]
The limit appears to be 1.

(c) $\lim_{x\to\pi/2} \frac{\sec x}{\tan x} = \lim_{x\to\pi/2} \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \lim_{x\to\pi/2} \frac{1}{\sin x} = 1$

55. Find c such that $\lim_{x\to 0} f(x) = c$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{9x - 3\sin 3x}{5x^3}$$
$$= \lim_{x\to 0} \frac{9-9\cos 3x}{15x^2}$$
$$= \lim_{x\to 0} \frac{27\sin 3x}{30x}$$
$$= \lim_{x\to 0} \frac{81\cos 3x}{30} = \frac{81}{30} = \frac{27}{10}$$

Thus, $c = \frac{27}{10}$. This works since $\lim_{x\to 0} f(x) = c = f(0)$, so f is continuous.

56. $f(x)$ is defined at $x \neq 0$. $\lim_{x\to 0} f(x)$ leads to the indeterminate form 0^0.

$$\ln|x|^x = x\ln|x| = \frac{\ln|x|}{\frac{1}{x}}$$

$$\lim_{x\to 0} \frac{\ln|x|}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -x = 0$$

$$\lim_{x\to 0} |x|^x = \lim_{x\to 0} e^{x\ln|x|} = e^0 = 1$$

Thus, f has a removable discontinuity at $x = 0$. Extend the definition of f by letting $f(0) = 1$.

57. (a) The limit leads to the indeterminate form 1^∞.

Let $f(k) = \left(1+\frac{r}{k}\right)^{kt}$.

$$\ln f(k) = kt\ln\left(1+\frac{r}{k}\right) = \frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}$$

$$\lim_{k\to\infty}\frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty}\frac{t\left(-\frac{r}{k^2}\right)\left(1+\frac{r}{k}\right)^{-1}}{-\frac{1}{k^2}}$$

$$= \lim_{k\to\infty}\frac{rt}{1+\frac{r}{k}} = \frac{rt}{1} = rt$$

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} = A_0\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{kt}$$

$$= A_0\lim_{k\to\infty} e^{\ln f(k)}$$

$$= A_0 e^{rt}$$

(b) Part (a) shows that as the number of compoundings per year increases toward infinity, the limit of interest compounded k times per year is interest compounded continuously.

58. (a) For $x \neq 0$, $\frac{f'(x)}{g'(x)} = \frac{1}{1} = 1.$

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = 1$$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{2}{1} = 2$$

(b) This does not contradict L'Hôpital's Rule since $\lim_{x\to 0} f(x) = 2$ and $\lim_{x\to 0} g(x) = 1$.

59. (a) $A(t) = \int_0^t e^{-x}dx$

$$= \left[-e^{-x}\right]_0^t = e^{-t}+1$$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}(-e^{-t}+1)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{e^t}+1\right) = 1$$

(b) $V(t) = \pi\int_0^t (e^{-x})^2\,dx$

$$= \pi\int_0^t e^{-2x}\,dx$$

$$= \pi\left[-\frac{1}{2}e^{-2x}\right]_0^t$$

$$= \pi\left(-\frac{1}{2}e^{-2t}+\frac{1}{2}\right)$$

$$= \frac{\pi}{2}\left(-e^{-2t}+1\right)$$

$$\lim_{t\to\infty}\frac{V(t)}{A(t)} = \lim_{t\to\infty}\frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{-t}+1}$$

$$= \frac{\frac{\pi}{2}(1)}{1} = \frac{\pi}{2}$$

(c) $$\lim_{t\to 0^+}\frac{V(t)}{A(t)} = \lim_{t\to 0^+}\frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{-t}+1}$$

$$= \lim_{t\to 0^+}\frac{\frac{\pi}{2}(2e^{-2t})}{e^{-t}}$$

$$= \frac{\frac{\pi}{2}(2)}{1} = \pi$$

60. (a)

x	$f(x)$
0.1	0.04542
0.01	0.00495
0.001	0.00050
0.0001	0.00005

The limit appears to be 0.

(b) $\lim_{x\to 0}\frac{\sin x}{1+2x} = \frac{0}{1} = 0$

L'Hôpital's Rule is not applied here because the limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, since the denominator has limit 1.

61. **(a)** $f(x) = e^{x\ln(1+1/x)}$

$1+\dfrac{1}{x} > 0$ when $x < -1$ or $x > 0$

Domain: $(-\infty, -1) \cup (0, \infty)$

(b) The form is 0^{-1}, so $\lim\limits_{x\to -1^-} f(x) = \infty$

(c) $$\lim_{x\to-\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to-\infty} \frac{\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{\left(-\frac{1}{x^2}\right)\left(1+\frac{1}{x}\right)^{-1}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{x\ln(1+1/x)} = e$$

62. False. Need $g'(a) \neq 0$. Consider $f(x) = \sin^2 x$ and $g(x) = x^2$ with $a = 0$. Here $\lim\limits_{x\to0} f'(x) = \lim\limits_{x\to0} g'(x) = 0$.

63. False. The limit is 1.

64. C. $\lim\limits_{x\to0} \dfrac{x}{\tan x} = \dfrac{1}{\sec^2 x} = \dfrac{1}{1} = 1$

65. D. $\lim\limits_{x\to1} \dfrac{1-\frac{1}{x}}{1-\frac{1}{x^2}} = \lim\limits_{x\to1} \dfrac{\frac{1}{x^2}}{\frac{2}{x^3}} = \lim\limits_{x\to1} \dfrac{x^3}{2x^2} = 1/2$

66. B. $\lim\limits_{x\to\infty} \dfrac{\log_2 x}{\log_3 x} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x\ln 2}}{\frac{1}{x\ln 3}} = \dfrac{\ln 3}{\ln 2}$

67. E. $\ln\left(1+\dfrac{1}{x}\right)^{3x} = 3x\ln\left(1+\dfrac{1}{x}\right) = \dfrac{\ln\left(1+\frac{1}{x}\right)}{1/(3x)}$

$$\lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x}\right)}{1/(3x)}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{\left(1+\frac{1}{x}\right)}\cdot\frac{-1}{x^2}}{-1/(3x^2)}\right)$$

$$= \lim_{x\to\infty} \frac{3}{\left(1+\frac{1}{x}\right)} = \frac{3}{1+0} = 3$$

$$\lim_{x\to\infty} e^{\ln f(x)} = e^3$$

68. Possible answers:

(a) $f(x) = 7(x-3)$; $g(x) = x-3$

$$\lim_{x\to3}\frac{f(x)}{g(x)} = \lim_{x\to3}\frac{7(x-3)}{x-3} = \lim_{x\to3}\frac{7}{1} = 7$$

(b) $f(x) = (x-3)^2$; $g(x) = x-3$

$$\lim_{x\to3}\frac{f(x)}{g(x)} = \lim_{x\to3}\frac{(x-3)^2}{x-3} = \lim_{x\to3}\frac{2(x-3)}{1} = 0$$

(c) $f(x) = x-3$; $g(x) = (x-3)^3$

$$\lim_{x\to3}\frac{f(x)}{g(x)} = \lim_{x\to3}\frac{x-3}{(x-3)^3} = \lim_{x\to3}\frac{1}{3(x-3)^2} = \infty$$

69. Answers may vary.

(a) $f(x) = 3x+1$; $g(x) = x$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{3x+1}{x} = \lim_{x\to\infty}\frac{3}{1} = 3$$

(b) $f(x) = x+1$; $g(x) = x^2$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x+1}{x^2} = \lim_{x\to\infty}\frac{1}{2x} = 0$$

(c) $f(x) = x^2$; $g(x) = x+1$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x^2}{x+1} = \lim_{x\to\infty}\frac{2x}{1} = \infty$$

70. **(a)** Because the difference in the numerator is so small compared to the values being subtracted, any calculator or computer with limited precision will give the incorrect result that $1-\cos x^6$ is 0 for even moderately small values of x. For example, at $x = 0.1$, $\cos x^6 = 0.9999999999995$ (13 places), so on a 10-place calculator, $\cos x^6 = 1$ and $1 - \cos x^6 = 0$.

(b) Same reason as in part (a) applies.

(c) $$\lim_{x\to0}\frac{1-\cos x^6}{x^{12}} = \lim_{x\to0}\frac{6x^5\sin x^6}{12x^{11}}$$

$$= \lim_{x\to0}\frac{\sin x^6}{2x^6}$$

$$= \lim_{x\to0}\frac{6x^5\cos x^6}{12x^5}$$

$$= \lim_{x\to0}\frac{\cos x^6}{2} = \frac{1}{2}$$

(d) The graph and/or table on a grapher show the value of the function to be 0 for x-values moderately close to 0, but the limit is 1/2. The calculator is giving unreliable information because there is significant round-off error in computing values of this function on a limited precision device.

71. (a) $f'(x)=3x^2,\ g'(x)=2x-1$

$f(1)-f(-1)=2,\ g(1)-g(-1)=-2$

$$\frac{3c^2}{2c-1}=\frac{2}{-2}$$
$$3c^2=-2c+1$$
$$3c^2+2c-1=0$$
$$(3c-1)(c+1)=0$$
$$c=\frac{1}{3} \text{ or } c=-1$$

The value of c that satisfies the property is $c=\frac{1}{3}$.

(b) $f'(x)=-\sin x,\ g'(x)=\cos x$

$$f\left(\frac{\pi}{2}\right)-f(0)=-1,\ g\left(\frac{\pi}{2}\right)-g(0)=1$$
$$\frac{-\sin c}{\cos c}=\frac{-1}{1}$$
$$\tan c=1$$
$$c=\tan^{-1}1=\frac{\pi}{4} \text{ on } \left(0,\frac{\pi}{2}\right)$$

72. (a) $\ln f(x)^{g(x)}=g(x)\ln f(x)$

$$\lim_{x\to c}(g(x)\ln f(x))=\left(\lim_{x\to c}g(x)\right)\left(\lim_{x\to c}\ln f(x)\right)$$
$$=\infty(-\infty)=-\infty$$
$$\lim_{x\to c}f(x)^{g(x)}=\lim_{x\to c}e^{\ln f(x)^{g(x)}}=e^{-\infty}=0$$

(b) $\lim_{x\to c}\left(g(x)\ln f(x)\right)=\left(\lim_{x\to c}g(x)\right)\left(\lim_{x\to c}\ln f(x)\right)$

$$=(-\infty)(-\infty)=\infty$$
$$\lim_{x\to c}f(x)^{g(x)}=\lim_{x\to c}e^{\ln f(x)^{g(x)}}=e^{\infty}=\infty$$

Quick Quiz Sections 8.1 and 8.2

1. C. $\lim_{x\to 0}\frac{(x+1)^{4/3}-(4/3)x-1}{x^2}$

$$=\lim_{x\to 0}\frac{(4/3)(x+1)^{1/3}-(4/3)}{2x}$$
$$=\lim_{x\to 0}\frac{(4/9)(x+1)^{-2/3}}{2}$$
$$=\frac{2}{9}$$

2. D. $\lim_{x\to 0^+}(3x^{2x})$

$$\ln x^{2x}=2x\ln x=\frac{2\ln x}{1/x}$$
$$\lim_{x\to 0^+}\frac{2\ln x}{1/x}=\lim_{x\to 0^+}\frac{2/x}{-1/x^2}=\lim_{x\to 0^+}(-2x)=0$$
$$\lim_{x\to 0^+}3e^{\ln f(x)}=3e^0=3$$

3. B. $\lim_{x\to 2}\frac{\int_2^x \sin t\,dt}{x^2-4}$

$$=\lim_{x\to 2}\frac{-\cos x+\cos 2}{x^2-4}$$
$$=\lim_{x\to 2}\frac{\sin x}{2x}=\frac{\sin 2}{4}$$

4. (a) $\left(\frac{1/2}{-4}\right)^{1/3}=-\frac{1}{2}$

$$\frac{-4}{-1/2}=8$$

(b) $-\frac{1}{2}$

(c) $a_n=8\left(-\frac{1}{2}\right)^{n-1}=(-1)^{n-1}(2^{4-n})$

(d) $a_n=\left(-\frac{1}{2}\right)a_{n-1}$

Section 8.3 Relative Rates of Growth (pp. 453–458)

Exploration 1 Comparing Rates of Growth as $x\to\infty$

1. $\lim_{x\to\infty}\frac{a^x}{x^2}=\lim_{x\to\infty}\frac{(\ln a)(a^x)}{2x}=\lim_{x\to\infty}\frac{(\ln a)^2a^x}{2}=\infty$, so a^x grows faster than x^2 as $x\to\infty$.
2. $\lim_{x\to\infty}\frac{3^x}{2^x}=\lim_{x\to\infty}1.5^x=\infty$
3. $\lim_{x\to\infty}\frac{a^x}{b^x}=\lim_{x\to\infty}\left(\frac{a}{b}\right)^x=\infty$ because $\frac{a}{b}>1$.

Quick Review 8.3

1. $\lim_{x\to\infty}\frac{\ln x}{e^x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{e^x}=0$
2. $\lim_{x\to\infty}\frac{e^x}{x^3}=\lim_{x\to\infty}\frac{e^x}{3x^2}=\lim_{x\to\infty}\frac{e^x}{6x}=\lim_{x\to\infty}\frac{e^x}{6}=\infty$
3. $\lim_{x\to-\infty}\frac{x^2}{e^{2x}}=\infty$
4. $\lim_{x\to\infty}\frac{x^2}{e^{2x}}=\lim_{x\to\infty}\frac{2x}{2e^{2x}}=\lim_{x\to\infty}\frac{2}{4e^{2x}}=0$
5. $-3x^4$
6. $\frac{2x^3}{x}=2x^2$