54. **(a)** Average temp

$$=\frac{1}{14-6}\int_6^{14}\left(80-10\cos\left(\frac{\pi t}{12}\right)\right)dt=87°F.$$

(b) $F(t)=80-10\cos\left(\frac{\pi t}{12}\right)\geq 78$ for

$5.2308694\leq t\leq 18.769131.$

Store these two values as A and B.

(c) Cost $=0.05\int_A^B\left(80-10\cos\left(\frac{\pi t}{12}\right)-78\right)dt\approx 5.10$

The cost was about \$5.10.

55. **(a)** $\int_9^{17}\frac{15600}{(t^2-24t+160)}dt\approx 6004$ people.

(b) $15\int_9^{17}\frac{15600}{(t^2-24t+160)}dt$

$+11\int_{17}^{23}\frac{15600}{(t^2-24t+160)}dt\approx 104{,}048$

dollars

(c) $H'(17)=E(17)-L(17)\approx -380$ people. $H(17)$ is the number of people in the park at 5:00, and $H'(17)$ is the rate at which the number of people in the park is changing at 5:00.

(d) When $H'(t)=E(t)-L(t)=0$; that is, at $t=15.795$

Chapter 8
Sequences, L'Hôpital's Rule, and Improper Integrals

Section 8.1 Sequences (pp. 435–443)

Quick Review 8.1

1. $f(5)=\frac{5}{5+3}=\frac{5}{8}$

2. $f(-2)=\frac{-2}{-2+3}=-2$

3. $-2+(3-1)(1.5)=1$

4. $-7+(5-1)(3)=5$

5. $1.5(2^{4-1})=12$

6. $-2(1.5^{3-1})=-4.5$

7. $\lim_{x\to\infty}\frac{5x^3+2x^2}{3x^4+16x^2}=\lim_{x\to\infty}\frac{5x^3}{3x^4}=0$

8. Recall from Chapter 2 that $\lim_{t\to 0}\frac{\sin t}{t}=1.$

Let $t=3x$. Then $t\to 0$ as $x\to 0$, and

$$\lim_{x\to 0}\frac{\sin(3x)}{x}=\lim_{t\to 0}\frac{\sin t}{t/3}=3\cdot\lim_{t\to 0}\frac{\sin t}{t}=3\cdot 1=3$$

9. Let $t=\frac{1}{x}$. Then $t\to 0$ as $x\to\infty$, and

$$\lim_{x\to\infty}\left(x\sin\left(\frac{1}{x}\right)\right)=\lim_{t\to 0}\left(\frac{1}{t}\cdot\sin(t)\right)=\lim_{t\to 0}\frac{\sin t}{t}=1$$

10. $\lim_{x\to\infty}\frac{2x^3+x^2}{x+1}=\lim_{x\to\infty}\frac{2x^3}{x}=\lim_{x\to\infty}2x^2=\infty$, or does not exist.

Section 8.1 Exercises

1. $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},\frac{6}{7};\frac{50}{51}$

2. $2,\frac{5}{2},\frac{8}{3},\frac{11}{4},\frac{14}{5},\frac{17}{6};\frac{149}{50}$

3. $2,\frac{9}{4},\frac{64}{27},\frac{625}{256},\frac{7776}{3125}\approx 2.48832,$

$\frac{117649}{46656}\approx 2.521626;\left(\frac{51}{50}\right)^{50}\approx 2.691588$

4. $-2,-2,0,4,10,18;\ 2350$

5. $3,1,-1,-3;\ -11$

6. $-2,-1,0,1;\ 5$

7. $2,4,8,16;\ 256$

8. $10,11,12.1,13.31;\ 19.487171\approx 10(1.1)^7$

9. $1,1,2,3;\ 21$

10. $-3,2,-1,1;\ 2$

11. **(a)** 3

(b) $a+7d=-2+7(3)=19$

(c) $a_n=a_{n-1}+3$

(d) $a_n=-2+(n-1)(3)=3n-5$

12. **(a)** -2

(b) $a+7d=15+7(-2)=1$

(c) $a_n=a_{n-1}-2$

(d) $a_n=15+(n-1)(-2)=-2n+17$

13. **(a)** $\frac{1}{2}$

(b) $a+7d=1+7\left(\frac{1}{2}\right)=\frac{9}{2}$

(c) $a_n=a_{n-1}+\frac{1}{2}$

(d) $a_n=1+(n-1)\left(\frac{1}{2}\right)=\frac{(n+1)}{2}$

14. **(a)** 0.1

(b) $a+7d=3+7(0.1)=3.7$

(c) $a_n=a_{n-1}+0.1$

(d) $a_n=3+(n-1)(0.1)=0.1n+2.9$

15. (a) $\frac{1}{2}$

(b) $8\left(\frac{1}{2}\right)^8 = 0.03125$

(c) $a_n = \left(\frac{1}{2}\right)a_{n-1}$

(d) $a_n = 8\left(\frac{1}{2}\right)^{n-1} = 2^{4-n}$

16. (a) 1.5

(b) $(1)(1.5)^8 \approx 25.6289$

(c) $a_n = (1.5)a_{n-1}$

(d) $a_n = (1)(1.5)^{n-1} = (1.5)^{n-1}$

17. (a) -3

(b) $(-3)(-3)^8 = (-3)^9 = -19,683$

(c) $a_n = -3a_{n-1}$

(d) $a_n = (-3)(-3)^{n-1} = (-3)^n$

18. (a) -1

(b) $(5)(-1)^8 = 5$

(c) $a_n = -a_{n-1}$

(d) $a_n = 5(-1)^{n-1}$

19. $d = \frac{7-(-2)}{3} = 3$

$a_1 = -2-3 = -5$

$a_n = a_{n-1} + 3$ for all $n \geq 2$

20. $d = \frac{-3-5}{4} = -2$

$a_1 = 5-(-2)(4) = 13$

$a_n = 13 + (n-1)(-2) = -2n+15$

21. $r = \left(\frac{3010000}{3010}\right)^{1/3} = 10$

$a_1 = \frac{3010}{10^3} = 3.01$

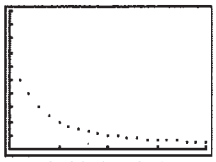
$a_n = 3.01(10)^{n-1}, n \geq 1$

22. $r = \left(\frac{16}{-1/2}\right)^{1/5} = -2$

$a_1 = \frac{-1/2}{-2} = 1/4$

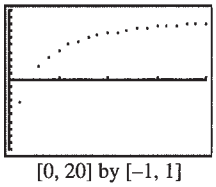
$a_n = \frac{1}{4}(-2)^{n-1} = (-1)^{n-1} \cdot (2)^{n-3},\ n \geq 1$

23.



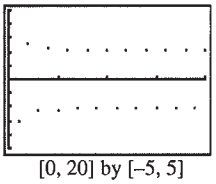
[0, 20] by [0, 1]

24.



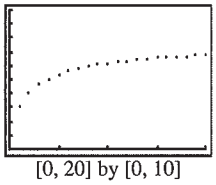
[0, 20] by [–1, 1]

25.



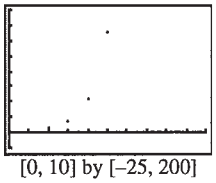
[0, 20] by [–5, 5]

26.



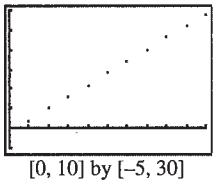
[0, 20] by [0, 10]

27.



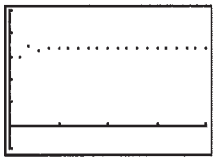
[0, 10] by [–25, 200]

28.



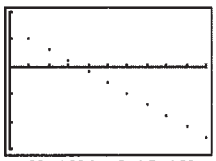
[0, 10] by [–5, 30]

29.



[0, 20] by [–1, 5]

30.



[0, 10] by [–15, 10]

31. $\lim_{n\to\infty} \frac{3n+1}{n} = \lim_{n\to\infty}(3) + \lim_{n\to\infty}\left(\frac{1}{n}\right) = 3-0 = 3$

converges, 3

32. $\lim_{n\to\infty}\frac{2n}{n+3}=\lim_{n\to\infty}\frac{2n}{n}=2$

converges, 2

33. $\lim_{n\to\infty}\frac{2n^2-n-1}{5n^2+n+2}=\lim_{n\to\infty}\frac{2n^2}{5n^2}=\frac{2}{5}$

converges, 2/5

34. $\lim_{n\to\infty}\frac{n}{n^2+1}=\lim_{n\to\infty}\frac{n}{n^2}=\lim_{n\to\infty}\frac{1}{n}=0$

converges, 0

35. $n=2k, \lim_{n\to\infty}(-1)^n\frac{n-1}{n+3}=1$

$n=2k-1, \lim_{n\to\infty}(-1)^n\frac{n-1}{n+3}=-1$

diverges

36. $n=2k, \lim_{n\to\infty}(-1)^n\frac{n+1}{n^2+1}=\lim_{n\to\infty}(-1)^n\frac{n}{n^2}=0$

$n=2k-1, \lim_{n\to\infty}(-1)^n\frac{n+1}{n^2+1}=\lim_{n\to\infty}(-1)^n\frac{n}{n^2}=0$

converges, 0

37. $\lim_{n\to\infty}(1.1)^n=\infty$

diverges

38. $\lim_{n\to\infty}(0.9)^n=0$

converges, 0

39. Let $x=\frac{1}{n}$. Then $x\to 0$ as $n\to\infty$, and

$$\lim_{n\to\infty}\left(n\sin\left(\frac{1}{n}\right)\right)=\lim_{n\to\infty}\frac{\sin(1/n)}{1/n}=\lim_{x\to 0}\frac{\sin x}{x}=1$$

converges, 1

40. Diverges. The first terms of the sequence are: $1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, \ldots$

The pattern repeats forever, so the sequence does not get close to any one number.

41. $\lim_{n\to\infty}\frac{\sin n}{n}=0$, since

$-\frac{1}{n}\le\frac{\sin n}{n}\le\frac{1}{n}$ for $n\ge 1$, and

$\lim_{n\to\infty}\left(-\frac{1}{n}\right)=\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$

42. $\lim_{n\to\infty}\left(\frac{1}{2^n}\right)=0$, since

$0\le\frac{1}{2^n}\le\frac{1}{n}$ for $n\ge 1$, and

$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$

43. $\lim_{n\to\infty}\left(\frac{1}{n!}\right)=0$, since

$0\le\frac{1}{n!}\le\frac{1}{n}$ for $n\ge 1$, and

$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$

44. $\lim_{n\to\infty}\left(\frac{\sin^2 n}{2^n}\right)=0$, since

$0\le\frac{\sin^2 n}{n}\le\frac{1}{n}$ for $n\ge 1$, and

$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$

45. Graph (b)

46. Graph (c)

47. Table (d)

48. Table (a)

49. False. If the sequence is increasing, the terms will eventually become positive. Consider the sequence with nth term $a_n=-5+2(n-1)$. Here $a=-5$, $a_2=-3$, $a_3=-1$, and $a_4=1$.

50. True. $a_1>0$, $r=\frac{a_2}{a_1}>0$, and

$a_n=a_1 r^{n-1}>0$ for all $n\ge 2$.

51. C. $d=\frac{5-(-1)}{2}=3$

$a_6=-1+3(5)=14$

52. E. $r=\frac{1.25}{2.5}=\frac{1}{2}$

$a_1=\frac{2.5}{1/2}=5$

53. D. Let $x=\frac{3\pi}{n}$. Then $x\to 0$ as $n\to\infty$, and

$$\begin{aligned}\lim_{n\to\infty}\left(n\sin\left(\frac{3\pi}{n}\right)\right)&=\lim_{x\to 0}\left(\frac{3\pi}{x}\cdot\sin(x)\right)\\&=3\pi\cdot\lim_{x\to 0}\frac{\sin x}{x}\\&=3\pi\cdot 1=3\pi\end{aligned}$$

54. E. $n=2k$, $\lim_{n\to\infty}\left((-1)^n\frac{3n-1}{n+2}\right)=3$

$n=2k-1, \lim_{n\to\infty}\left((-1)^n\frac{3n-1}{n+2}\right)=-3$

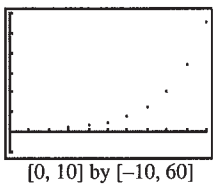
55. (a) Draw a segment from the center of the circle to each vertex of the polygon, forming n isosceles triangles. The vertex angle in each triangle is $2\pi/n$. An altitude to the base divides the isosceles triangle into two right triangles with hypotenuse 1. In each of these triangles, the side opposite the angle of measure π/n has length $\sin(\pi/n)$. It follows that each side of the polygon has length $2\sin(\pi/n)$ and the total perimeter is $2n\sin(\pi/n)$.

(b) Let $x = \frac{\pi}{n}$. Then $x \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty}\left(2n\sin\left(\frac{\pi}{n}\right)\right) = \lim_{x\to 0}\left(2\cdot\frac{\pi}{x}\cdot\sin(x)\right)$$
$$= 2\pi\cdot\lim_{x\to 0}\frac{\sin x}{x} = 2\pi\cdot 1 = 2\pi$$

56. (a) $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$
$a_1 = 1;\ a_2 = 1$
$a_3 = a_1 + a_2 = 1 + 1 = 2$
$a_4 = a_2 + a_3 = 1 + 2 = 3$
Continue in this fashion to get all of the first ten terms:
1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(b)



[0, 10] by [−10, 60]

57. $a_n = ar^{n-1}$ implies that $\log a_n = \log a + (n-1)\log r$. Thus $\{\log a_n\}$ is an arithmetic sequence with first term $\log a$ and common difference $\log r$.

58. $a_n = a + (n-1)d$ implies that $10^{a_n} = 10^{a+(n-1)d} = 10^a(10^d)^{n-1}$. Thus $\{10^{a_n}\}$ is a geometric sequence with first term 10^a and common ratio 10^d.

59. Given $\varepsilon > 0$ choose $M = \frac{1}{\varepsilon}$. Then $\left|\frac{1}{n} - 0\right| < \varepsilon$ if $n > M$.

Section 8.2 L'Hôpital's Rule (pp. 444–452)

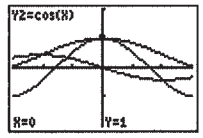
Exploration 1 Exploring L'Hôpital's Rule Graphically

1. $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$

2. The two graphs suggest that $\lim_{x\to 0}\frac{y_1}{y_2} = \lim_{x\to 0}\frac{y_1'}{y_2'}$.

3. $y_5 = \frac{x\cos x - \sin x}{x^2}$. The graphs of y_3 and y_5 clearly show that l'Hôpital's Rule does not say that $\lim_{x\to 0}\frac{y_1}{y_2}$ is equal to

$$\lim_{x\to 0}\left(\frac{y_1}{y_2}\right)'.$$



[−3, 3] by [−2, 2]

Quick Review 8.2

1.

x	$\left(1+\frac{0.1}{x}\right)^x$
1	1.1000
10	1.1046
100	1.1051
1000	1.1052
10,000	1.1052
1,000,000	1.1052

As $x \to \infty$, $\left(1+\frac{0.1}{x}\right)^x$ approaches 1.1052.

2.

x	$x^{1/(\ln x)}$
0.1	2.7183
0.01	2.7183
0.001	2.7183
0.0001	2.7183
0.00001	2.7183

As $x \to 0^+$, $x^{1/(\ln x)}$ approaches 2.7183.

3.

x	$\left(1-\frac{1}{x}\right)^x$
−1	0.5
−0.1	0.78679
−0.01	0.95490
−0.001	0.99312
−0.0001	0.99908
−0.00001	0.99988
−0.000001	0.99999

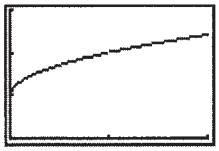
As $x \to 0^-$, $\left(1-\frac{1}{x}\right)^x$ approaches 1.

4.

x	$\left(1+\frac{1}{x}\right)^x$
−1.1	13.981
−1.01	105.77
−1.001	1007.9
−1.0001	10010

As $x \to -1^-$, $\left(1+\frac{1}{x}\right)^x$ goes to ∞.

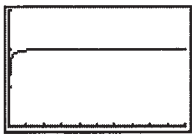
5.



[0, 2] by [0, 3]

As $t \to 1$, $\frac{t-1}{\sqrt{t}-1}$ approaches 2.

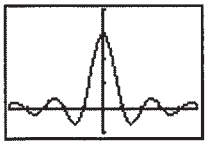
6.



[0, 500] by [0, 3]

As $x \to \infty$, $\frac{\sqrt{4x^2+1}}{x+1}$ approaches 2.

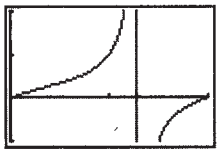
7.



[−5, 5] by [−1, 4]

As $x \to 0$, $\frac{\sin 3x}{x}$ approaches 3.

8.



[0, π] by [−1, 2]

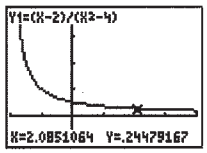
As $\theta \to \frac{\pi}{2}$, $\frac{\tan\theta}{2+\tan\theta}$ approaches 1.

9. $y = \frac{1}{h}\sin h = \frac{\sin h}{h}$

10. $y = (1+h)^{1/h}$

Section 8.2 Exercises

1. $\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right)$ appears to be about $\frac{1}{4}$;

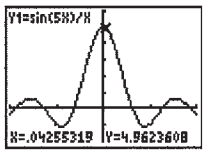


[−2, 4] by [−1, 4]

By L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right) = \lim_{x\to 2}\frac{1}{2x} = \frac{1}{4}$$

2. $\lim_{x\to 0}\left(\frac{\sin(5x)}{x}\right)$ appears to be about 5;

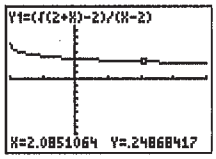


[−2, 2] by [−2, 6]

By L'Hôpital's Rule:

$$\lim_{x\to 0}\left(\frac{\sin(5x)}{x}\right) = \lim_{x\to 0}\frac{5\cos(5x)}{1} = 5\cos(0) = 5$$

3. $\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\right)$ appears to be about $\frac{1}{4}$;



[−2, 4] by [−1, 1]

By L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\right) = \lim_{x\to 2}\left(\frac{\frac{1}{2}(2+x)^{-1/2}}{1}\right) = \frac{1}{2}(2+2)^{-1/2}$$

$$= \frac{1}{2\sqrt{4}} = \frac{1}{4}$$