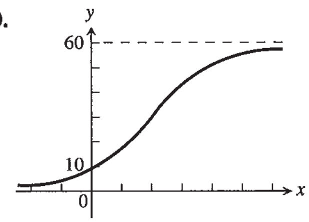
7. As $x \to -\infty$, $-0.1x \to \infty$, and $e^{-0.1x} \to \infty$, so

$$\lim_{x\to-\infty} \frac{60}{1+5e^{-0.1x}} = 0$$

8. $y(0) = \dfrac{60}{1+5\,e^{(-0.1(0))}} = 10$

9. From problems 6 and 7, the two horizontal asymptotes are $y = 0$ and $y = 60$.

10.



Section 6.5 Exercises

1. $A(x-4)+B(x)=x-12$

$x=4,\quad 4B=-8$

$B=-2$

$x=1,\ A(1-4)+(-2)(1)=1-12$

$A=3$

2. $A(x-2)+B(x+3)=2x+16$

$x=2,\ B(2+3)=2(2)+16$

$5\,B=20$

$B=4$

$x=-3,\ A(-3-2)=2(-3)+16$

$-5\,A=10$

$A=-2$

3. $A(x+5)+B(x-2)=16-x$

$x=-5,\ B(-5-2)=16-(-5)$

$-7B=21$

$B=-3$

$x=2,\ A(2+5)=16-2$

$7A=14$

$A=2$

4. $A(x+3)+B(x-3)=3$

$x=-3,\ B(-3-3)=3$

$-6B=3$

$B=-1/2$

$x=3,\ A(3+3)=3$

$6A=3$

$A=1/2$

5. See problem 1.

$$\int \frac{x-12}{x^2-4}dx = \int\left(\frac{3}{x}+\frac{-2}{x-4}\right)dx$$
$$= 3\,\ln\,|x| - 2\,\ln\,|x-4| + C$$
$$= \ln\left(\frac{|x|^3}{(x-4)^2}\right)+C$$

6. See problem 2.

$$\int \frac{2x+16}{x^2+x-6}dx = \int\left(\frac{-2}{x+3}+\frac{4}{x-2}\right)dx$$
$$= -2\,\ln\,|x+3| + 4\,\ln\,|x-2| + C$$
$$= \ln\left(\frac{(x-2)^4}{(x+3)^2}\right)+C$$

7.
$$\begin{array}{r} 2x \\ x^2-4\overline{)\,2x^3\qquad} \\ \underline{2x^3-8x} \\ 8x \end{array}$$

$$\int\left(2x+\frac{8x}{x^2-4}\right)dx$$

$u=x^2-4$

$du=2x\ dx$

$$x^2+4\int\frac{du}{u} = x^2+4\,\ln\,|u|+C$$
$$= x^2+\ln(x^2-4)^4+C$$

8.
$$\begin{array}{r} 1 \\ x^2-9\overline{)\,x^2-6} \\ \underline{x^2-9} \\ 3 \end{array}$$

$$\int 1+\frac{3}{x^2-9}dx$$
$$= x+\int\frac{A}{x+3}+\frac{B}{x-3}dx$$

$A(x-3)+B(x+3)=3$

$x=3,\ B(3+3)=3$

$B=1/2$

$x=-3,\ A(-3-3)=3$

$A=-1/2$

$$x+\int\frac{-1/2}{x+3}+\frac{1/2}{x-3}dx$$
$$= x+\ln\sqrt{\frac{|x-3|}{|x+3|}}+C$$

9. $2\displaystyle\int\frac{dx}{x^2+1} = 2\tan^{-1}x+C$

10. $3\displaystyle\int\frac{dx}{x^2+9} = \tan^{-1}\left(\frac{x}{3}\right)+C$

11. $\displaystyle\int\frac{7}{2x^2-5x-3}dx$

$$\frac{A}{2x+1}+\frac{B}{x-3}=\frac{7}{(2x+1)(x-3)}$$

$A(x-3)+B(2x+1)=7$

$x=3,\ B(2(3)+1)=7$

$B=1$

$x=-1/2,\ A(-1/2-3)=7$

$A=-2$

$$\int\left(\frac{-2}{2x+1}+\frac{1}{x-3}\right)dx$$
$$= \ln\left|\frac{x-3}{2x+1}\right|+C$$

12. $\int \frac{1-3x}{3x^2-5x-3}dx$

$\frac{A}{3x-2}+\frac{B}{x-1}=\frac{1-3x}{(3x-2)(x-1)}$

$A(x-1)+B(3x-2)=1-3x$

$x=1, \quad B(3(1)-2)=1-3(1)$

$B=-2$

$x=\frac{2}{3}, \quad A\left(\frac{2}{3}-1\right)=1-3\left(\frac{2}{3}\right)$

$-\frac{1}{3}A=-1$

$A=3$

$\int\left(\frac{3}{3x-2}+\frac{-2}{x-1}\right)dx$

$=\ln|3x-2|-2\ln|x-1|+C$

$=\ln\left|\frac{3x-2}{(x-1)^2}\right|+C$

13. $\int \frac{8x-7}{2x^2-x-3}$

$\frac{A}{x+1}+\frac{B}{2x-3}=\frac{8x-7}{(x+1)(2x-3)}$

$A(2x-3)+B(x+1)=8x-7$

$x=\frac{3}{2}, \quad B\left(\frac{3}{2}+1\right)=8\left(\frac{3}{2}\right)-7$

$\frac{5}{2}B=5$

$B=2$

$x=-1, \quad A(2x-3)=8x-7$

$A(-2-3)=8-7$

$-5A=-15$

$A=3$

$\int\left(\frac{3}{x+1}+\frac{2}{2x-3}\right)dx$

$=3\ln|x+1|+\ln|2x-3|+C$

$=\ln\left(|x+1|^3|2x-3|\right)+C$

14. $\int \frac{5x+14}{x^2+7x}dx$

$\frac{A}{x}+\frac{B}{x+7}=\frac{5x+14}{x(x+7)}$

$A(x+7)+Bx=5x+14$

$x=-7, \quad -7B=5(-7)+14$

$-7B=-21$

$B=3$

$x=0, \quad A(0+7)=5(0)+14$

$7A=14$

$A=2$

$\int\left(\frac{2}{x}+\frac{3}{x+7}\right)dx$

$=2\ln|x|+3\ln|x+7|+C$

$=\ln\left(x^2|x+7|^3\right)+C$

15. $\int dy=\int \frac{2x-6}{x^2-2x}dx$

$\frac{A}{x}+\frac{B}{x-2}=\frac{2x-6}{x(x-2)}$

$A(x-2)+Bx=2x-6$

$x=2, \quad 2B=2(2)-6$

$2B=-2$

$B=-1$

$x=0, A(0-2)=2(0)-6$

$-2A=-6$

$A=3$

$\int\left(\frac{3}{x}+\frac{-1}{x-2}\right)dx$

$y=3\ln|x|-\ln|x-2|+C$

$y=\ln\left|\frac{x^3}{x-2}\right|+C$

16. $\int du=\int \frac{2}{x^2-1}dx$

$\frac{A}{x+1}+\frac{B}{x-1}=\frac{2}{(x+1)(x-1)}$

$A(x-1)+B(x+1)=2$

$x=1, \quad B(1+1)=2$

$2B=2$

$B=1$

$x=-1, \quad A(-1-1)=2$

$-2A=2$

$A=-1$

$u=\int\left(\frac{-1}{x+1}+\frac{1}{x-1}\right)dx$

$u=-\ln|x+1|+\ln|x-1|+C$

$u=\ln\left|\frac{x-1}{x+1}\right|+C$

17. $\int F'(x)\,dx=\int \frac{2}{x^3-x}dx$

$\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}=\frac{2}{x(x+1)(x-1)}$

$A(x+1)(x-1)+Bx(x-1)+Cx(x+1)=2$

$x=1, \quad 2C=2$

$C=1$

$x=-1, \quad 2B=2$

$B=1$

$x=0, \quad -A=2$

$A=-2$

$\int\left(\frac{-2}{x}+\frac{1}{x+1}+\frac{1}{x-1}\right)dx$

$F(x)=-2\ln|x|+\ln|x+1|+\ln|x-1|+C$

$F(x)=\ln\left(\frac{|x^2-1|}{x^2}\right)+C$

18. $\int G'(t)\,dt = \int \frac{2t^3}{t^3 - t}\,dt$

$$\begin{array}{r} 2 \\ t^3 - t\overline{\big)\,2t^3} \\ \underline{2t^3 - 2t} \\ 2t \end{array}$$

$$= \int 2 + \frac{2t}{t^3 - t}\,dt$$

$$= 2t + \int \frac{2}{t^2 - 1}\,dt$$

$$\frac{A}{t-1} + \frac{B}{t+1} = \frac{2}{(t-1)(t+1)}$$

$$A(t+1) + B(t-1) = 2$$

$$t = -1,\quad B(-1-1) = 2$$
$$-2B = 2$$
$$B = -1$$

$$t = 1,\qquad A(1+1) = 2$$
$$2A = 2$$
$$A = 1$$

$$G(t) = 2t + \int \left(\frac{1}{(t-1)} + \frac{-1}{(t+1)} \right) dt$$

$$= 2t + \ln|t-1| - \ln|t+1| + C$$

$$= 2t + \ln\left|\frac{t-1}{t-1}\right| + C$$

19. $\int \frac{2x}{x^2 - 4}\,dx$

$$u = x^2 - 4$$
$$du = 2x\,dx$$
$$\int \frac{du}{u} = \ln u + C = \ln|x^2 - 4| + C$$

20. $\int \frac{4x - 3}{2x^2 - 3x + 1}\,dx$

$$u = 2x^2 - 3x + 1$$
$$du = (4x - 3)\,dx$$
$$\int \frac{du}{u} = \ln u + C$$
$$= \ln|2x^2 - 3x + 1| + C$$

21. $\int \frac{x^2 + x - 1}{x^2 - x}\,dx$

$$\begin{array}{r} 1 \\ x^2 - x\overline{\big)\,x^2 + x - 1} \\ \underline{x^2 - x} \\ 2x - 1 \end{array}$$

$$\int \left(1 + \frac{2x - 1}{x^2 - x} \right) dx$$

$$u = x^2 - x$$
$$du = (2x - 1)\,dx$$
$$x + \int \frac{du}{u} = x + \ln u + C$$
$$= x + \ln|x^2 - x| + C$$

22. $\int \frac{2x^3}{x^2 - 1}\,dx$

$$\begin{array}{r} 2x \\ x^2 - 1\overline{\big)\,2x^3} \\ \underline{2x^3 - 2x} \\ 2x \end{array}$$

$$\int \left(2x + \frac{2x}{x^2 - 1} \right) dx$$

$$u = x^2 - 1$$
$$du = 2x\,dx$$
$$x^2 + \int \frac{du}{u}$$
$$= x^2 + \ln u + C$$
$$= x^2 + \ln|x^2 - 1| + C$$

23. **(a)** 200 individuals.

(b) 100 individuals.

(c) $\frac{dP(100)}{dt} = 0.006(100)(200 - 100)$
$= 60$ individuals per year.

24. **(a)** 700 individuals.

(b) 350 individuals.

(c) $\frac{dP(350)}{dt} = 0.0008(350)(700 - 350)$
$= 98$ individuals per year.

25. **(a)** 1200 individuals.

(b) 600 individuals.

(c) $\frac{dP(600)}{dt} = 0.0002(600)(1200 - 600)$
$= 72$ individuals per year.

26. **(a)** 5000 individuals.

(b) 2500 individuals.

(c) $\frac{dP(2500)}{dt} = 10^{-5}(2500)(5000 - 2500)$
$= 62.5$ individuals per year.

27. $\frac{dP}{dt} = .006\ P(200 - P)$

$$\int \frac{dP}{P(200 - P)} = \int .006\,dt$$

$$\frac{A}{P} + \frac{B}{200 - P} = \frac{1}{P(200 - P)}$$

$$A(200 - P) + BP = 1$$

$$P = 200,\quad 200\,B = 1$$
$$B = 0.005$$

$$P = 0,\ A(200 - 0) = 1$$
$$200A = 1$$
$$A = 0.005$$

$$\int \left(\frac{0.005}{P} + \frac{0.005}{200 - P} \right) dP = 0.006t$$

$$\int \left(\frac{1}{P} + \frac{1}{200 - P} \right) dP = 1.2t$$

$$\ln P - \ln(200 - P) = 1.2t + C$$

27. Continued

$$\ln\left(\frac{200-P}{P}\right)=-1.2t-C$$

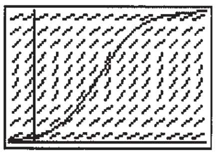
$$\frac{200}{P}-1=e^{-1.2t}\ e^{-c}$$

$$\frac{200}{P}=1+e^{-1.2t}\ e^{-c}$$

$$\frac{200}{8}=1+e^{-1.2(0)}\ e^{-c}$$

$$e^{-c}=24$$

$$P=\frac{200}{1+24\,e^{-1.2t}}$$



[−1, 7] by [0, 200]

28. $\dfrac{dP}{dt}=0.0008\ P(700-P)$

$$\int\frac{dP}{P(700-P)}=\int 0.0008\,dt$$

$$\frac{A}{P}+\frac{B}{700-P}=\frac{1}{P(700-P)}$$

$$A(700-P)+BP=1$$

$$P=700,\quad 700B=1$$

$$B=1/700$$

$$P=0,\quad A(700-0)=1$$

$$A=1/700$$

$$\int\left(\frac{1/700}{P}+\frac{1/700}{700-P}\right)dP=0.0008t$$

$$\int\left(\frac{1}{P}+\frac{1}{700-P}\right)dP=0.56t$$

$$\ln P-\ln(700-P)=0.56t+C$$

$$\ln\left(\frac{700-P}{P}\right)=-0.56t-C$$

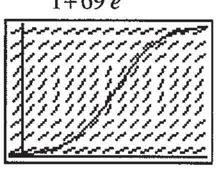
$$\frac{700}{P}-1=e^{-0.56t}\ e^{-c}$$

$$\frac{700}{P}=1+e^{-0.56t}\ e^{-c}$$

$$\frac{700}{10}=1+e^{-0.56(0)}\ e^{-c}$$

$$e^{-c}=69$$

$$P=\frac{700}{1+69\,e^{-0.56t}}$$



[−1, 15] by [0, 700]

29. $\dfrac{dP}{dt}=0.0002\,P(1200-P)$

$$\int\frac{dP}{P(1200-P)}=\int 0.0002\,dt$$

$$\frac{A}{P}+\frac{B}{1200-P}=\frac{1}{P(1200-P)}$$

$$A(1200-P)+BP=1$$

$$P=1200,\quad 1200B=1$$

$$B=1/1200$$

$$P=0,\ A(1200-0)=1$$

$$1200A=1$$

$$A=1/1200$$

$$\int\left(\frac{1/1200}{P}+\frac{1/1200}{700-P}\right)dP=0.0002t$$

$$\int\left(\frac{1}{P}+\frac{1}{1200-P}\right)dP=0.24t$$

$$\ln P-\ln(1200-P)=0.24t+C$$

$$\ln\left(\frac{1200-P}{P}\right)=-0.24\,t-C$$

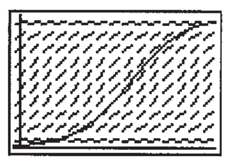
$$\frac{1200}{P}-1=e^{-0.24t}\ e^{-c}$$

$$\frac{1200}{P}=1+e^{-0.24t}\ e^{-c}$$

$$\frac{1200}{20}=1+e^{-0.24(0)}\ e^{-c}$$

$$e^{-c}=59$$

$$P=\frac{1200}{1+59\,e^{-0.24t}}$$



[−1, 30] by [0, 1200]

30. $\dfrac{dP}{dt}=10^{-5}\,P(5000-P)$

$$\int\frac{dP}{P(5000-P)}=\int 10^{-5}\,dt$$

$$\frac{A}{P}+\frac{B}{5000-P}=\frac{1}{P(5000-P)}$$

$$A(5000-P)+BP=1$$

$$P=5000,\quad 5000B=1$$

$$B=0.0002$$

$$P=0,\ A(5000-0)=1$$

$$5000A=1$$

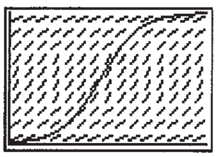
$$A=0.0002$$

$$\int\left(\frac{0.0002}{P}+\frac{0.0002}{5000-P}\right)dP=10^{-5}t$$

$$\int\left(\frac{1}{P}+\frac{1}{5000-P}\right)dP=0.05t$$

30. Continued

$$\ln P - \ln(5000 - P) = 0.05t + C$$
$$\ln\left(\frac{5000 - P}{P}\right) = -0.05t - C$$
$$\frac{5000}{P} - 1 = e^{-0.05t}e^{-c}$$
$$\frac{5000}{P} = 1 + e^{-0.05t}e^{-c}$$
$$\frac{5000}{50} = 1 + e^{-0.05(0)}e^{-c}$$
$$e^{-c} = 99$$
$$P = \frac{5000}{1 + 99e^{-0.05t}}$$



[–1, 200] by [0, 5000]

31. (a) $P(t) = \frac{1000}{1 + e^{4.8 - 0.7t}}$
$$= \frac{1000}{1 + e^{4.8 - 0.7t}}$$
$$= \frac{M}{1 + Ae^{-Mkt}}$$

This is a logistic growth model with $M = 1000$ and $k = \frac{0.7}{1000} = 0.0007$.

(b) $P(0) = \frac{1000}{1 + e^{4.8}} \approx 8$

Initially there are 8 rabbits.

32. (a) $P(t) = \frac{200}{1 + e^{5.3 - t}}$
$$= \frac{200}{1 + e^{5.3}e^{-t}}$$
$$= \frac{M}{1 + Ae^{-Mkt}}$$

This is a logistic growth model with $M = 200$ and $k = \frac{1}{200} = 0.05$.

(b) $P(0) = \frac{200}{1 + e^{5.3}} \approx 1$

Initially 1 student has the measles.

33. (a) $\frac{dP}{dt} = 0.0015P(150 - P)$
$$= kP(M - P)$$

Thus, $k = 0.0015$ and $M = 150$.
$$P = \frac{M}{1 + Ae^{-Mkt}}$$
$$= \frac{150}{1 + Ae^{-0.225t}}$$

Initial condition: $P(0) = 6$
$$6 = \frac{150}{1 + Ae^{0}}$$
$$1 + A = 25$$
$$A = 24$$

Formula: $P = \frac{150}{1 + 24e^{-0.225t}}$

(b)
$$100 = \frac{150}{1 + 24e^{-0.225t}}$$
$$1 + 24e^{-0.225t} = \frac{3}{2}$$
$$24e^{-0.225t} = \frac{1}{2}$$
$$e^{-0.225t} = \frac{1}{48}$$
$$-0.225t = -\ln 48$$
$$t = \frac{\ln 48}{0.225} \approx 17.21 \text{ weeks}$$
$$125 = \frac{150}{1 + 24e^{-0.225t}}$$
$$1 + 24e^{-0.225t} = \frac{6}{5}$$
$$24e^{-0.225t} = \frac{1}{5}$$
$$e^{-0.225t} = \frac{1}{120}$$
$$-0.225t = -\ln 120$$
$$t = \frac{\ln 120}{0.225} \approx 21.28$$

It will take about 17.21 weeks to reach 100 guppies, and about 21.28 weeks to reach 125 guppies.

34. (a) $\frac{dP}{dt} = 0.0004P(250 - P)$
$$= kP(M - P)$$

Thus, $k = 0.0004$ and $M = 250$.
$$P = \frac{M}{1 + Ae^{-Mkt}}$$
$$= \frac{250}{1 + Ae^{-0.1t}}$$

Initial condition: $P(0) = 28$, where $t = 0$ represents the year 1970.
$$28 = \frac{250}{1 + Ae^{0}}$$
$$28(1 + A) = 250$$
$$A = \frac{250}{28} - 1 = \frac{111}{14} \approx 7.9286$$

Formula: $P(t) = \frac{250}{1 + 111e^{-0.1t}/14}$, or approximately
$$P(t) = \frac{250}{1 + 7.9286e^{-0.1t}}$$

34. Continued

(b) The population $P(t)$ will round to 250 when $P(t) \geq 249.5$.

$$249.5 = \frac{250}{1+111e^{-0.1t}/14}$$

$$249.5\left(1+\frac{111e^{-0.1t}}{14}\right) = 250$$

$$\frac{(249.5)(111e^{-0.1t})}{14} = 0.5$$

$$e^{-0.1t} = \frac{14}{55{,}389}$$

$$-0.1t = \ln\frac{14}{55{,}389}$$

$$t = 10(\ln 55{,}389 - \ln 14) \approx 82.8$$

It will take about 83 years.

35. $\dfrac{dP}{dt} = kP(M-P)$

$$\int \frac{dP}{P(M-P)} = \int k\,dt$$

$$\frac{Q}{P} + \frac{R}{M-P} = \frac{1}{P(M-P)}$$

$$Q(M-P) + RP = 1$$

$$P = 0,\ MQ = 1$$

$$Q = \frac{1}{M}$$

$$P = M,\ MR = 1$$

$$R = \frac{1}{M}$$

$$\int\left(\frac{\frac{1}{M}}{P} + \frac{\frac{1}{M}}{(M-P)}\right)dP = kt + C$$

$$\int\left(\frac{1}{P} + \frac{1}{(M-P)}\right)dP = Mkt + C$$

$$\ln\left(\frac{M-P}{P}\right) = -Mkt - C$$

$$\frac{M}{P} - 1 = e^{-Mkt}e^{-C}$$

$$\frac{M}{P} = 1 + e^{-Mkt}A$$

$$P = \frac{M}{1+Ae^{-Mkt}}$$

36. (a) $\dfrac{dP}{dt} = k(M-P)$

$$\int \frac{dP}{M-P} = \int k\,dt$$

$$-\ln(M-P) = kt + C$$

$$M - P = e^{-kt}\,e^{-c}$$

Let $e^{-c} = A$

then $P = M - Ae^{-kt}$

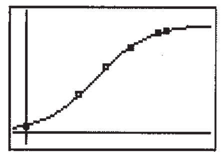
(b) $\lim_{t\to\infty} P(t) = M - Ae^{-k\infty} = M$

(c) When $t = 0$.

(d) This curve has no inflection point. If the initial population is greater than M, the curve is always concave up and approaches $y = M$ asymptotically from above. If the initial population is smaller than M, the curve is always concave down and approaches $y = M$ asymptotically from below.

37. (a) The regression equation is

$$P = \frac{232739.9}{1+14.582e^{-0.101t}}$$



[–5, 70] by [–24000, 260000]

(b) $\lim_{t\to\infty} \dfrac{232739.9}{1+14.582e^{-0.101t}} = \dfrac{232739.9}{1+0} \approx 232{,}740$ people.

(c) $225{,}000 = \dfrac{232739.9}{1+14.582e^{-0.101t}}$

$$14.582e^{-0.101t} = \frac{232739.9}{225{,}000} - 1$$

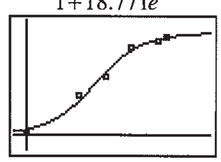
$$e^{-0.101t} = .0024$$

$$t = \frac{-6.05}{-0.101} \approx 60 \text{ or in 2010.}$$

(d) $\dfrac{dP}{dt} = kP(M-P) = (4.352\times 10^{-7})P(232739.9 - P)$.

38. (a) The regression equation is

$$P = \frac{458791.8}{1+18.771e^{-0.113t}}$$



(b) $\lim_{t\to\infty} \dfrac{458791.8}{1+18.771e^{-0.113(t)}} = \dfrac{458791.8}{1+0} \approx 458{,}792$

(c) $450{,}000 = \dfrac{458791.8}{1+18.771e^{-0.113t}}$

$$18.771e^{-0.113t} = \frac{458791.8}{450{,}000} - 1$$

$$e^{-0.113t} = 0.001$$

$$t = \frac{-6.87}{-0.113} = 60 \text{ or in 2010.}$$

(d) $\dfrac{dP}{dt} = kP(M-P) = (2.4626\times 10^{-7})P(458791.8 - P)$

39. False. It does look exponential, but it resembles the solution to $\dfrac{dP}{dt} = kP(100-10) = (90k)P$.

40. True. The graph will be a logistic curve with $\lim_{t\to\infty} P(t) = 100$ and $\lim_{t\to-\infty} P(t) = 0$.

41. D. $\frac{600}{2} = 300.$

42. B. $M = 0.9$, so at most 90% of the population will be infected. The remaining 10% will not be infected.

43. D. $\int_2^3 \frac{3}{(x-1)(x+2)}dx$

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{3}{(x-1)(x+2)}$$
$$A(x+2) + B(x-1) = 3$$
$$x = -2, \quad B(-2-1) = 3$$
$$-3B = 3$$
$$B = -1$$
$$x = 1, \quad A(1+2) = 3$$
$$3A = 3$$
$$A = 1$$
$$\int\left(\frac{1}{x-1} + \frac{-1}{x+2}\right)dx$$
$$= \ln\left(\frac{x-1}{x+2}\right)\Bigg|_2^3 = \ln\left(\frac{8}{5}\right)$$

44. B.

45. **(a)** Note that $k > 0$ and $M > 0$, so the sign of $\frac{dP}{dt}$ is the sameas the sign of $(M-P)(P-m)$. For $m < P < M$, both $M-P$ and $P-m$are positive, so the product is positive. For $P < m$or $P > M$, the expressions $M-P$ and $P-m$ have opposite signs, so the product is negative.

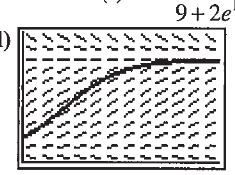
(b) $\frac{dP}{dt} = \frac{k}{M}(M-P)(P-m)$

$$\frac{dP}{dt} = \frac{k}{1200}(1200-P)(P-100)$$
$$\frac{1200}{(1200-P)(P-100)}\frac{dP}{dt} = k$$
$$\frac{1100}{(1200-P)(P-100)}\frac{dP}{dt} = \frac{11}{12}k$$
$$\frac{(P-100)+(1200-P)}{(1200-P)(P-100)}\frac{dP}{dt} = \frac{11}{12}k$$
$$\left(\frac{1}{1200-P} + \frac{1}{P-100}\right)\frac{dP}{dt} = \frac{11}{12}k$$
$$\int\left(\frac{1}{1200-P} + \frac{1}{P-100}\right)dP = \int\frac{11}{12}k\,dt$$
$$-\ln|1200-P| + \ln|P-100| = \frac{11}{12}kt + C$$
$$\ln\left|\frac{P-100}{1200-P}\right| = \frac{11}{12}kt + C$$
$$\frac{P-100}{1200-P} = \pm e^C e^{11kt/12}$$
$$\frac{P-100}{1200-P} = Ae^{11kt/12}$$
$$P - 100 = 1200Ae^{11kt/12} - APe^{11kt/12}$$
$$P(1 + Ae^{11kt/12}) = 1200Ae^{11kt/12} + 100$$
$$P = \frac{1200Ae^{11kt/12} + 100}{1 + Ae^{11kt/12}}$$

(c) $300 = \frac{1200Ae^0 + 100}{1 + Ae^0}$

$$300(1+A) = 1200A + 100$$
$$300 - 100 = 1200A - 300A$$
$$200 = 900A$$
$$A = \frac{2}{9}$$
$$P(t) = \frac{1200(2/9)e^{11kt/12} + 100}{1 + (2/9)e^{11kt/12}}$$
$$P(t) = \frac{1200(2)e^{11kt/12} + 100(9)}{9 + 2e^{11kt/12}}$$
$$P(t) = \frac{300(8e^{11kt/12} + 3)}{9 + 2e^{11kt/12}}$$

(d)



[0, 75] by [0, 1500]

Note that the slope field is given by

$$\frac{dP}{dt} = \frac{0.1}{1200}(1200-P)(P-100).$$

(e) $\frac{dP}{dt} = \frac{k}{M}(M-P)(P-m)$

$$\frac{M}{(M-P)(P-m)}\frac{dP}{dt} = k$$
$$\frac{M}{M-m}\frac{M-m}{(M-P)(P-m)}\frac{dP}{dt} = k$$
$$\frac{(P-m)+(M-P)}{(M-P)(P-m)}\frac{dP}{dt} = \frac{M-m}{M}k$$
$$\left(\frac{1}{M-P} + \frac{1}{P-m}\right)\frac{dP}{dt} = \frac{M-m}{M}k$$
$$\int\left(\frac{1}{M-P} + \frac{1}{P-m}\right)dP = \int\frac{M-m}{M}k\,dt$$
$$-\ln|M-P| + \ln|P-m| = \frac{M-m}{M}kt + C$$
$$\ln\left|\frac{P-m}{M-P}\right| = \frac{M-m}{M}kt + C$$
$$\frac{P-m}{M-P} = \pm e^C e^{(M-m)kt/M}$$
$$\frac{P-m}{M-P} = Ae^{(M-m)kt/M}$$
$$P - m = (M-P)Ae^{(M-m)kt/M}$$
$$P(1 + Ae^{(M-m)kt/M}) = AMe^{(M-m)kt/M} + m$$
$$P = \frac{AMe^{(M-m)kt/M} + m}{1 + Ae^{(M-m)kt/M}}$$
$$P(0) = \frac{AMe^0 + m}{1 + Ae^0} = \frac{AM + m}{1 + A}$$

45. Continued

(e)
$$P(0)(1+A)=AM+m$$
$$A(P(0)-M)=m-P(0)$$
$$A=\frac{m-P(0)}{P(0)-M}=\frac{P(0)-m}{M-P(0)}$$

Therefore, the solution to the differential equation is

$$P=\frac{AMe^{(M-m)kt/M}+m}{1+Ae^{(M-m)kt/M}} \quad \text{where } A=\frac{P(0)-m}{M-P(0)}.$$

46. (a) Let $u=\frac{x}{a}$; then $x=au$, $\quad dx=a\,du$

$$\int\frac{dx}{a^2+x^2}=\int\frac{a\,du}{a^2+a^2u^2}=\frac{a}{a^2}\int\frac{du}{1+u^2}$$
$$=\frac{1}{a}\tan^{-1}(u)+C$$
$$=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

(b) Let $u=\frac{x}{a}$; then $x=au$, $\quad dx=a\,du$

$$\int\frac{dx}{a^2-x^2}=\int\frac{a\,du}{a^2-a^2u^2}=\frac{a}{a^2}\int\frac{du}{1-u^2}$$
$$=-\frac{1}{a}\int\frac{du}{u^2-1}=-\frac{1}{a}\int\frac{1}{(u+1)(u-1)}du$$

$$\frac{A}{u+1}+\frac{B}{u-1}=\frac{1}{(u+1)(u-1)}$$
$$A(u-1)+B(u+1)=1$$
$$u=-1,\quad A(-2)=1$$
$$A=-1/2$$
$$u=1,\quad B(2)=1$$
$$B=1/2$$

$$-\frac{1}{a}\int\left(\frac{-1/2}{u+1}+\frac{1/2}{u-1}\right)du=\frac{1}{2a}\int\left(\frac{1}{u+1}-\frac{1}{u-1}\right)du$$
$$=\frac{1}{2a}\ln\left|\frac{u+1}{u-1}\right|+C$$
$$=\frac{1}{2a}\ln\left|\frac{\frac{x}{a}+1}{\frac{x}{a}-1}\right|+C$$
$$=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C$$

(c) Let $u=a+x$, $\quad du=dx$

$$\int\frac{dx}{(a+x)^2}=\int\frac{du}{u^2}=-\frac{1}{u}+C=-\frac{1}{x+a}+C$$

47. (a) $\int\frac{5x}{(x+3)^2}dx$

$$\frac{A}{x+3}+\frac{B}{(x+3)^2}=\frac{5x}{(x+3)^2}$$
$$A(x+3)+B=5x$$
$$x=-3,\quad B=-15$$
$$A(x+3)-15=5x$$
$$x=0,\quad A(3)-15=0$$
$$A=5$$

$$\int\left(\frac{5}{x+3}-\frac{15}{(x+3)^2}\right)dx=5\ln|x+3|+\frac{15}{x+3}+C$$

(b) $\int\frac{5x}{(x+3)^3}dx$

$$\frac{A}{(x+3)}+\frac{B}{(x+3)^2}+\frac{C}{(x+3)^3}=\frac{5x}{(x+3)^3}$$
$$A(x+3)^2+B(x+3)+C=5x$$
$$x=-3,\quad C=-15$$
$$A(x+3)^2+B(x+3)-15=5x$$
$$x=0,\quad 9A+3B-15=0,\quad B=5+3A$$
$$x=1,\quad 16A+4B-15=5,\quad B=5+4A$$
$$5+3A=5+4A$$
$$A=0$$
$$B=5+3(0)=5$$

$$\int\left(\frac{5}{(x+3)^2}-\frac{15}{(x+3)^3}\right)dx=-\frac{5}{x+3}+\frac{15}{2(x+3)^2}+C$$

48. (a) This is true since

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}=\frac{A(x-1)^2+B(x-1)+C}{(x-1)^3}$$

(b) $A(x-1)^2+B(x-1)+C=x^2+3x+5$

$$x=0,\quad A-B+C=5$$
$$x=1,\quad C=9$$
$$x=2,\quad A+B+C=15$$

Then

$$A-B=-4$$
$$A+B=6$$
$$2A=2$$
$$A=1$$
$$B=5$$

(c)
$$\int\left(\frac{1}{x-1}+\frac{5}{(x-1)^2}+\frac{9}{(x-1)^3}\right)dx$$
$$=\ln|x-1|-\frac{5}{x-1}-\frac{9}{2(x-1)^2}+C$$

Quick Quiz

1. C. $y = y_0 e^{kt}$

$$t=1, \quad 2 = y_0 e^k$$
$$t=5, \quad 3 = y_0 e^{5k}$$
$$\frac{3}{2} = \frac{y_0 e^{5k}}{y_0 e^k} = e^{4k}$$
$$k = \frac{\ln(3/2)}{4}$$
$$y_0 = 2e^{-\frac{\ln(3/2)}{4}} = 1.807$$
$$t=8, \quad y = 1.807e^{\left(\frac{\ln(3/2)}{4}\cdot 8\right)} = 4.066$$

2. C. $F(x) = \int_a^x \cos(t^2)dt$

$$F(1) = 0 \Rightarrow a = 1$$
$$F(x) = \int_1^x \cos(t^2)dt$$
$$F(5) = \int_1^5 \cos(t^2)dt = -0.293$$

[Use NINT $(\cos(x^2), x, 1, 5)$ to evaluate the integral.]

3. A. $\int \frac{dx}{(x-1)(x+3)}$

$$\frac{A}{x-1} + \frac{B}{x+3} = \frac{1}{(x-1)(x+3)}$$
$$A(x+3) + B(x-1) = 1$$
$$x = -3, \quad -4B = 1$$
$$B = -1/4$$
$$x = 1, \quad 4A = 1$$
$$A = 1/4$$
$$\int\left(\frac{1/4}{x-1} + \frac{-1/4}{x+3}\right)dx$$
$$= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$$

4.
$$\frac{dP}{dt} = \frac{P}{5}\left(\frac{10-P}{10}\right)$$
$$\int \frac{dP}{P(10-P)} = \int \frac{1}{50}dt$$
$$\frac{A}{P} + \frac{B}{10-P} = \frac{1}{P(10-P)}$$
$$A(10-P) + BP = 1$$
$$P = 10, \quad 10B = 1$$
$$B = 0.1$$
$$P = 0, \quad 10A = 1$$
$$A = 0.1$$
$$\int\left(\frac{0.1}{P} + \frac{0.1}{10-P}\right)dP = \frac{1}{50}t + C$$
$$\ln\left|\frac{10-P}{P}\right| = -1/5t - C$$
$$P = \frac{10}{1+e^{-1/5t}e^{-C}}$$
$$P(0) = 3 = \frac{10}{1+Ae^{-1/5(0)}}$$
$$A = 2.33$$
$$\lim_{t\to\infty} P(t) = \frac{10}{1+2.33e^{-1/5(t)}} = 10$$

(b) $P(0) = 20 = \frac{10}{1+Ae^{-1/5(0)}}$

$$A = -0.5$$
$$\lim_{t\to\infty} P(t) = \frac{10}{1+-0.5e^{-1/5(t)}} = 10$$

(c) Separate the variables.

$$\frac{dY}{Y} = \frac{1}{5}\left(1 - \frac{t}{10}\right)dt$$
$$\ln Y = \frac{t}{5} - \frac{t^2}{100} + C_1$$
$$Y = Ce^{t/5 - t^2/100} \quad \text{where } C = e^{C_1}$$
$$3 = Ce^0 \Rightarrow C = 3$$
$$Y = 3e^{t/5 - t^2/100}$$

(d) $\lim_{t\to\infty} 3e^{t/5-t^2 100} = \lim_{t\to\infty} \frac{3e^{t/5}}{e^{t^2/100}} = \lim_{t\to\infty} \frac{3e^{t/5}}{\left(e^{t/5}\right)^{t/20}} = 0$

Chapter 6 Review Exercises

1. $\int_0^{\pi/3} \sec^2\theta \, d\theta = \tan\theta\Big]_0^{\pi/3} = \tan\frac{\pi}{3} - \tan 0 = \sqrt{3}$

2.
$$\int_1^2\left(x + \frac{1}{x^2}\right)dx = \left[\frac{1}{2}x^2 - x^{-1}\right]_1^2$$
$$= \left(\frac{1}{2}(4) - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$$
$$= \frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2$$

3. Let $u = 2x+1$

$$du = 2dx$$
$$\frac{1}{2}du = dx$$
$$\int_0^1 \frac{36}{(2x+1)^3}dx = 18\int_1^3 \frac{1}{u^3}du$$
$$= 18\left(-\frac{1}{2}\right)u^{-2}\Big]_1^3 = -9\left(\frac{1}{9} - 1\right)$$
$$= -9\left(-\frac{8}{9}\right) = 8$$

4. Let $u = 1 - x^2$

$$du = -2x\,dx$$
$$-du = 2x\,dx$$
$$\int_{-1}^1 2x\sin(1-x^2)dx = -\int_0^0 \sin u\,du = 0$$