7. $$\begin{aligned} 1.1^t &= 10 \\ \ln 1.1^t &= \ln 10 \\ t \ln 1.1 &= \ln 10 \\ t &= \frac{\ln 10}{\ln 1.1} = \frac{1}{\log 1.1} \approx 24.159 \end{aligned}$$

8. $$\begin{aligned} e^{-2t} &= \frac{1}{4} \\ -2t &= \ln\left(\frac{1}{4}\right) \\ t &= -\frac{1}{2}\ln\left(\frac{1}{4}\right) = \frac{1}{2}\ln 4 = \ln 2 \end{aligned}$$

9. $$\begin{aligned} \ln(y+1) &= 2x-3 \\ y+1 &= e^{2x-3} \\ y &= -1+e^{2x-3} \end{aligned}$$

10. $$\begin{aligned} \ln|y+2| &= 3t-1 \\ |y+2| &= e^{3t-1} \\ y+2 &= \pm e^{3t-1} \\ y &= -2 \pm e^{3t-1} \end{aligned}$$

Section 6.4 Exercises

1. $$\begin{aligned} \int y\,dy &= \int x\,dx \\ \frac{y^2}{2} &= \frac{x^2}{2}+C \\ (2)^2 &= (1)^2 + C \\ C &= 3 \end{aligned}$$

$y=\sqrt{x^2+3}$, valid for all real numbers

2. $$\begin{aligned} \int y\,dy &= -\int x\,dx \\ \frac{y^2}{2} &= -\frac{x^2}{2}+C \\ (3)^2 &= -(4)^2 + C \\ C &= 25 \end{aligned}$$

$y=\sqrt{25-x^2}$, valid on the interval $(-5, 5)$

3. $$\begin{aligned} \int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\ \ln|y| &= \ln|x| + C \\ |y| &= |x| + C \\ 2 &= 2 + C \\ C &= 0 \end{aligned}$$

$y=x$, valid on the interval $(0, \infty)$

4. $$\begin{aligned} \int \frac{1}{y}\,dy &= \int 2x\,dx \\ \ln y &= x^2 + C \\ |y| &= e^{x^2+C} = e^C e^{x^2} \\ y &= Ae^{x^2} \\ 3 &= Ae^{0^2} \\ 3 &= A \end{aligned}$$

$y=3e^{x^2}$, valid for all real numbers

5. $$\begin{aligned} \int \frac{dy}{y+5} &= \int (x+2)\,dx \\ \ln|y+5| &= \frac{x^2}{2}+2x+C \\ |y+5| &= e^{x^2/2+2x+C} = e^C e^{x^2/2+2x} \\ y+5 &= \pm e^C e^{x^2/2+2x} = Ae^{x^2/2+2x} \\ y &= Ae^{x^2/2+2x} - 5 \end{aligned}$$

$y = 6e^{x^2/2+2x} - 5$, valid for all real numbers.

6. $$\begin{aligned} \int \frac{dy}{\cos^2 y} &= \int dx \\ \tan y &= x + C \\ \tan(0) &= 0 + C \\ C &= 0 \end{aligned}$$

$y = \tan^{-1} x$, valid for all real numbers.

7. $$\begin{aligned} \frac{dy}{dx} &= \cos x\, e^y e^{\sin x} \\ \int e^{-y}\,dy &= \int \cos x\, e^{\sin x}\,dx \\ -e^{-y} &= e^{\sin x} + C \\ -e^0 &= e^{\sin 0} + C \\ C &= -2 \end{aligned}$$

$y = -\ln(2e - e^{\sin x})$, valid for all real numbers.

8. $$\begin{aligned} \frac{dy}{dx} &= e^{-y}e^x \\ \int e^y\,dy &= \int e^x\,dx \\ e^y &= e^x + C \\ C &= e^2 - e^0 = e^2 - 1 \end{aligned}$$

$y = \ln(e^x + e^2 - 1)$, valid for all real numbers.

9. $$\begin{aligned} \int \frac{1}{y^2}\,dy &= \int -2x\,dx \\ -y^{-1} &= -x^2 + C \\ \frac{1}{.25} &= 1 + C \\ C &= 3 \end{aligned}$$

$y = \frac{1}{x^2+3}$, valid for all real numbers.

10. $$\begin{aligned} \frac{dy}{dx} &= \frac{4\sqrt{y}\ln x}{x} \\ \int \frac{dy}{\sqrt{y}} &= \int \frac{4\ln x}{x}\,dx \\ u &= \ln x \\ du &= \frac{1}{x}\,dx \\ 2\sqrt{y} &= \int 4u\,du \\ 2\sqrt{y} &= 2u^2 + C \\ y &= (\ln x)^4 + C \\ 1 &= (\ln e) + C \\ C &= 0 \end{aligned}$$

$y = (\ln x)^4$, valid on the interval $(0, \infty)$.

11. $y(t) = y_0 e^{kt}$

$y(t) = 100e^{1.5t}$

12. $y(t) = y_0 e^{kt}$

$y(t) = 200e^{-0.5t}$

13.
$$\begin{aligned} y(t) &= y_0 e^{kt} \\ y(t) &= 50e^{kt} \\ y(5) = 100 &= 50e^{5k} \\ 2 &= e^{5k} \\ \ln 2 &= 5k \\ k &= 0.2 \ln 2 \end{aligned}$$

Solution : $y(t) = 50e^{(0.2 \ln 2)t}$ or $y(t) = 50 \cdot 2^{0.2t}$

14.
$$\begin{aligned} y(t) &= y_0 e^{kt} \\ y(t) &= 60e^{kt} \\ y(10) = 30 &= 60e^{10k} \\ \frac{1}{2} &= e^{10k} \\ \ln \frac{1}{2} &= 10k \\ k &= 0.1 \ln \frac{1}{2} = -0.1 \ln 2 \end{aligned}$$

Solution: $y(t) = 60e^{-(0.1 \ln 2)t}$ or $y(t) = 60 \cdot 2^{-t/10}$

15. Doubling time:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 2000 &= 1000e^{0.086t} \\ 2 &= e^{0.086t} \\ \ln 2 &= 0.086t \\ t &= \frac{\ln 2}{0.086} \approx 8.06 \text{ yr} \end{aligned}$$

Amount in 30 years:

$$A = 1000e^{(0.086)(30)} \approx \$13{,}197.14$$

16. Annual rate:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 4000 &= 2000e^{(r)(15)} \\ 2 &= e^{15r} \\ \ln 2 &= 15r \\ r &= \frac{\ln 2}{15} \approx 0.0462 = 4.62\% \end{aligned}$$

Amount in 30 years:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ A &= 2000e^{[(\ln 2)/15](30)} \\ &= 2000e^{2 \ln 2} \\ &= 2000 \cdot 2^2 \\ &= \$8000 \end{aligned}$$

17. Initial deposit:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 2898.44 &= A_0 e^{(0.0525)(30)} \\ A_0 &= \frac{2898.44}{e^{1.575}} \approx \$600.00 \end{aligned}$$

Doubling time:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 1200 &= 600e^{0.0525t} \\ 2 &= e^{0.0525t} \\ \ln 2 &= 0.0525t \\ t &= \frac{\ln 2}{0.0525} \approx 13.2 \text{ years} \end{aligned}$$

18. Annual rate:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 10{,}405.37 &= 1200e^{(r)(30)} \\ \frac{104.0537}{12} &= e^{30r} \\ \ln \frac{104.0537}{12} &= 30r \\ r &= \frac{1}{30} \ln \frac{104.0537}{12} \approx 0.072 = 7.2\% \end{aligned}$$

Doubling time:

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 2400 &= 1200e^{0.072t} \\ 2 &= e^{0.072t} \\ \ln 2 &= 0.072t \\ t &= \frac{\ln 2}{0.072} \approx 9.63 \text{ years} \end{aligned}$$

19. (a) Annually:

$$\begin{aligned} 2 &= 1.0475^t \\ \ln 2 &= t \ln 1.0475 \\ t &= \frac{\ln 2}{\ln 1.0475} \approx 14.94 \text{ years} \end{aligned}$$

(b) Monthly:

$$\begin{aligned} 2 &= \left(1 + \frac{0.0475}{12}\right)^{12t} \\ \ln 2 &= 12t \ln \left(1 + \frac{0.0475}{12}\right) \\ t &= \frac{\ln 2}{12 \ln \left(1 + \frac{0.0475}{12}\right)} \approx 14.62 \text{ years} \end{aligned}$$

(c) Quarterly:

$$\begin{aligned} 2 &= \left(1 + \frac{0.0475}{4}\right)^{4t} \\ \ln 2 &= 4t \ln 1.011875 \\ t &= \frac{\ln 2}{4 \ln 1.011875} \approx 14.68 \text{ years} \end{aligned}$$

(d) Continuously:

$$\begin{aligned} 2 &= e^{0.0475t} \\ \ln 2 &= 0.0475t \\ t &= \frac{\ln 2}{0.0475} \approx 14.59 \text{ years} \end{aligned}$$

20. **(a)** Annually:

$$2 = 1.0825^t$$
$$\ln 2 = t \ln 1.0825$$
$$t = \frac{\ln 2}{\ln 1.0825} \approx 8.74 \text{ years}$$

(b) Monthly:

$$2 = \left(1 + \frac{0.0825}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.0825}{12}\right)$$
$$t = \frac{\ln 2}{12 \ln\left(1 + \frac{0.0825}{12}\right)} \approx 8.43 \text{ years}$$

(c) Quarterly:

$$2 = \left(1 + \frac{0.0825}{4}\right)^{4t}$$
$$\ln 2 = 4t \ln 1.020625$$
$$t = \frac{\ln 2}{4 \ln 1.020625} \approx 8.49 \text{ years}$$

(d) Continuously:

$$2 = e^{0.0825t}$$
$$\ln 2 = 0.0825t$$
$$t = \frac{\ln 2}{0.0825} \approx 8.40 \text{ years}$$

21. $\frac{dy}{dt} = -0.0077y$

$$\int \frac{1}{y}\, dy = \int -0.0077\, dt$$
$$\ln y = -0.0077t$$
$$t = \frac{\ln (1/2)}{-0.0077} = 90 \text{ years}$$

22. $\frac{dy}{dt} = -ky$

$$\int \frac{1}{y}\, dy = \int (-k)\, dt$$
$$\ln y = -kt$$
$$-\frac{\ln (1/2)}{65} = k$$
$$k = 0.01067$$

23. **(a)** Since there are 48 half-hour doubling times in 24 hours, there will be $2^{48} \approx 2.8 \times 10^{14}$ bacteria.

(b) The bacteria reproduce fast enough that even if many are destroyed there are still enough left to make the person sick.

24. Using $y = y_0 e^{kt}$, we have $10{,}000 = y_0 e^{3k}$ and $40{,}000 = y_0 e^{5k}$. Hence $\frac{40{,}000}{10{,}000} = \frac{y_0 e^{5k}}{y_0 e^{3k}}$, which gives $e^{2k} = 4$, or $k = \ln 2$. Solving $10{,}000 = y_0 e^{3 \ln 2}$, we have $y_0 = 1250$. There were 1250 bacteria initially. We could solve this more quickly by noticing that the population increased by a factor of 4, i.e. doubled twice, in 2 hrs, so the doubling time is 1 hr. Thus in 3 hrs the population would have doubled 3 times, so the initial population was $\frac{10{,}000}{2^3} = 1250$.

25. $0.9 = e^{-0.18t}$

$$\ln 0.9 = -0.18t$$
$$t = -\frac{\ln 0.9}{0.18} \approx 0.585 \text{ days}$$

26. **(a)** Half-life $= \frac{\ln 2}{k} = \frac{\ln 2}{0.005} \approx 138.6$ days

(b)
$$0.05 = e^{-0.005t}$$
$$\ln 0.05 = -0.005t$$
$$t = -\frac{\ln 0.05}{0.005} \approx 599.15 \text{ days}$$

The sample will be useful for about 599 days.

27. Since $y_0 = y(0) = 2$, we have:

$$y = 2e^{kt}$$
$$5 = 2e^{(k)(2)}$$
$$\ln 5 = \ln 2 + 2k$$
$$k = \frac{\ln 5 - \ln 2}{2} = 0.5 \ln 2.5$$

Function: $y = 2e^{(0.5 \ln 2.5)t}$ or $y \approx 2e^{0.4581t}$

28. Since $y_0 = y(0) = 1.1$, we have:

$$y = 1.1e^{kt}$$
$$3 = 1.1e^{(k)(-3)}$$
$$\ln 3 = \ln 1.1 - 3k$$
$$k = \frac{1}{3}(\ln 1.1 - \ln 3)$$

Function: $y = 1.1e^{(\ln 1.1 - \ln 3)t/3}$ or $y \approx 1.1e^{-0.3344t}$

29. At time $t = \frac{3}{k}$, the amount remaining is $y_0 e^{-kt} = y_0 e^{-k(3/k)} = y_0 e^{-3} \approx 0.0498 y_0$. This is less than 5% of the original amount, which means that over 95% has decayed already.

30. $T - T_s = (T_0 - T_s)\, e^{-kt}$

$$35 - 65 = (T_0 - 65)e^{-(k)(10)}$$
$$50 - 65 = (T_0 - 65)e^{-(k)(20)}$$

Dividing the first equation by the second, we have:

$$2 = e^{10k}$$
$$k = \frac{1}{10} \ln 2$$

Substituting back into the first equation, we have:

$$-30 = (T_0 - 65)e^{-[(\ln 2)/10](10)}$$
$$-30 = (T_0 - 65)\left(\frac{1}{2}\right)$$
$$-60 = T_0 - 65$$
$$5 = T_0$$

The beam's initial temperature is 5°F.

31. (a) First, we find the value of k.

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$60 - 20 = (90 - 20)e^{-(k)(10)}$$
$$\frac{4}{7} = e^{-10k}$$
$$k = -\frac{1}{10}\ln\frac{4}{7}$$

When the soup cools to 35°, we have:

$$35 - 20 = (90 - 20)e^{[(1/10)\ \ln\ (4/7)]t}$$
$$15 = 70e^{[(1/10)\ \ln\ (4/7)]t}$$
$$\ln\frac{3}{14} = \left(\frac{1}{10}\ln\frac{4}{7}\right)t$$
$$t = \frac{10\ln\left(\frac{3}{14}\right)}{\ln\left(\frac{4}{7}\right)} \approx 27.53 \text{ min}$$

It takes a total of about 27.53 minutes, which is an additional 17.53 minutes after the first 10 minutes.

(b) Using the same value of k as in part (a), we have:

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$35 - (-15) = [90 - (-15)]e^{[(1/10)\ \ln\ (4/7)]t}$$
$$50 = 105e^{[(1/10)\ \ln\ (4/7)]t}$$
$$\ln\frac{10}{21} = \left(\frac{1}{10}\ln\frac{4}{7}\right)t$$
$$t = \frac{10\ \ln\left(\frac{10}{21}\right)}{\ln\left(\frac{4}{7}\right)} \approx 13.26$$

It takes about 13.26 minutes

32. First, we find the value of k.
Taking "right now" as $t=0$, 60° above room temperature means $T_0 - T_s = 60$. Thus, we have

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$70 = 60e^{(-k)(-20)}$$
$$\frac{7}{6} = e^{20k}$$
$$k = \frac{1}{20}\ln\frac{7}{6}$$

(a) $T - T_s = (T_0 - T_s)\ e^{-kt} = 60e^{(-(1/20)\ln(7/6))(15)} \approx 53.45$

It will be about 53.45°C above room temperature.

(b) $T - T_s = (T_0 - T_s)\ e^{-kt} = 60e^{(-(1/20)\ln(7/6))(120)} \approx 23.79$

It will be about 23.79° above room temperature.

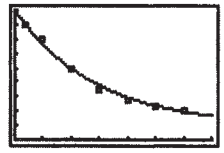
(c) $T - T_s = (T_0 - T_s)e^{-kt}$

$$10 = 60e^{(-(1/20)\ln\ (7/6))t}$$
$$\ln\frac{1}{6} = \left(-\frac{1}{20}\ln\frac{7}{6}\right)t$$
$$t = -\frac{20\ln(1/6)}{\ln(7/6)} \approx 232.47 \text{ min}$$

It will take about 232.47 min or 3.9 hr.

33. (a) $T - T_s = 79.47(0.932)^t$

(b) $T = 10 + 79.47(0.932)^t$



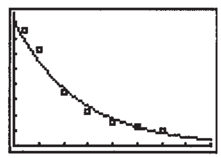
[0, 35] by [0, 90]

(c) Solving $T=12$ and using the exact values from the regression equation, we obtain $t \approx 52.5$ sec.

(d) Substituting $t=0$ into the equation we found in part (b), the temperature was approximately 89.47°C.

34. (a) Newton's Law of Cooling predicts that the difference between the probe temperature (T) and the surrounding temperature (T_s) is an exponential function of time, but in this case $T_s = 0$, so T is an exponential function of time.

(b) $T = 79.96 \times 0.9273^t$



[–0, 40] by [0, 86]

(c) At about 37 seconds.

(d) 76.96°C

35. Use $k = \frac{\ln 2}{5700}$ (see Example 3).

$$e^{-kt} = 0.445$$
$$-kt = \ln 0.445$$
$$t = -\frac{\ln 0.445}{k} = -\frac{5700\ln 0.445}{\ln 2} \approx 6658 \text{ years}$$

Crater Lake is about 6658 years old.

36. Use $k = \frac{\ln 2}{5700}$ (see Example 3).

(a) $e^{-kt} = 0.17$

$$-kt = \ln\ 0.17$$
$$t = -\frac{\ln 0.17}{k} = -\frac{5700\ \ln 0.17}{\ln 2} \approx 14{,}571 \text{ years}$$

The animal died about 14,571 years before A.D. 2000, in 12,571 B.C.

(b) $e^{-kt} = 0.18$

$$-kt = \ln\ 0.18$$
$$t = -\frac{\ln\ 0.18}{k} = -\frac{5700\ln 0.18}{\ln 2} \approx 14{,}101 \text{ years}$$

The animal died about 14,101 years before A.D. 2000, in 12,101 B.C.

36. Continued

(c) $e^{-kt} = 0.16$

$-kt = \ln 0.16$

$t = -\dfrac{\ln 0.16}{k} = -\dfrac{5700 \ln 0.16}{\ln 2} \approx 15{,}070$ years

The animal died about 15,070 years before A.D. 2000, in 13,070 B.C.

37. $\dfrac{1}{3} = e^{-kt}$

$k = \dfrac{\ln(1/3)}{5} = 0.22$

$-\dfrac{\ln(1/3)}{5} = -0.22$

$\dfrac{1}{2} = e^{-0.22t}$

$t = \dfrac{\ln(1/2)}{-0.22} = 3.15$ years

38. $3 = e^{rt}$

$r = \dfrac{\ln(3)}{10} = 0.11$

$4 = e^{0.11t}$

$t = \dfrac{\ln(4)}{0.11} = 12.60$ years

39. $y = y_0 e^{-kt}$

$800 = 1000e^{-(k)(10)}$

$0.8 = e^{-10k}$

$k = -\dfrac{\ln 0.8}{10}$

At $t = 10 + 14 = 24$ h:

$y = 1000e^{-(-\ln 0.8/10)24}$

$= 1000e^{2.4 \ln 0.8} \approx 585.4$ kg

About 585.4 kg will remain.

40. $0.2 = e^{-0.1t}$

$\ln 0.2 = -0.1t$

$t = -10 \ln 0.2 \approx 16.09$ yr

It will take about 16.09 years.

41. (a) $\dfrac{dp}{dh} = kp$

$\dfrac{dp}{p} = k\,dh$

$\int \dfrac{dp}{p} = \int k\,dh$

$\ln|p| = kh + C$

$e^{\ln|p|} = e^{kh+C}$

$|p| = e^C e^{kh}$

$p = Ae^{kh}$

Initial condition: $p = p_0$ when $h = 0$

$p_0 = Ae^0$

$A = p_0$

Solution: $p = p_0 e^{kh}$

Using the given altitude-pressure data, we have $p_0 = 1013$ millibars, so:

$p = 1013e^{kh}$

$90 = 1013e^{(k)(20)}$

$\dfrac{90}{1013} = e^{20k}$

$k = \dfrac{1}{20} \ln \dfrac{90}{1013} \approx -0.121\,\text{km}^{-1}$

Thus, we have $p \approx 1013e^{-0.121h}$

(b) At 50 km, the pressure is

$1013e^{((1/20)\ln(90/1013))(50)} \approx 2.383$ millibars.

(c) $900 = 1013e^{kh}$

$\dfrac{900}{1013} = e^{kh}$

$h = \dfrac{1}{k} \ln \dfrac{900}{1013} = \dfrac{20 \ln(900/1013)}{\ln(90/1013)} \approx 0.977$ km

The pressure is 900 millibars at an altitude of about 0.977 km.

42. By the Law of Exponential Change, $y = 100e^{-0.6t}$. At $t = 1$ hour, the amount remaining will be

$100e^{-0.6(1)} \approx 54.88$ grams.

43. (a) By the Law of Exponential Change, the solution is

$V = V_0 e^{-(1/40)t}$.

(b) $0.1 = e^{-(1/40)t}$

$\ln 0.1 = -\dfrac{t}{40}$

$t = -40 \ln 0.1 \approx 92.1$ sec

It will take about 92.1 seconds.

44. (a) $A(t) = A_0 e^t$

It grows by a factor of e each year.

(b) $3 = e^t$

$\ln 3 = t$

It will take $\ln 3 \approx 1.1$ yr.

(c) In one year your account grows from A_0 to $A_0 e$, so you can earn $A_0 e - A_0$, or $(e-1)$ times your initial amount. This represents an increase of about 172%.

45. (a) $90 = e^{(r)(100)}$

$\ln 90 = 100r$

$r = \dfrac{\ln 90}{100} \approx 0.045$ or 4.5%

(b) $131 = e^{(r)(100)}$

$\ln 131 = 100r$

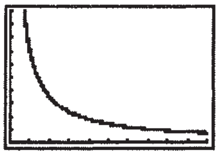
$r = \dfrac{\ln 131}{100} \approx 0.049$ or 4.9%

46. (a) $2y_0 = y_0 e^{rt}$

$2 = e^{rt}$

$\ln 2 = rt$

$t = \dfrac{\ln 2}{r}$

(b) 

[0, 0.1] by [0, 100]

(c) $\ln 2 \approx 0.69$, so the doubling time is $\dfrac{0.69}{r}$ which is almost the same as the rules.

(d) $\dfrac{70}{5} = 14$ years or $\dfrac{72}{5} = 14.4$ years

(e) $3y_0 = y_0 e^{rt}$

$3 = e^{rt}$

$\ln 3 = rt$

$t = \dfrac{\ln 3}{r}$

Since $\ln 3 \approx 1.099$, a suitable rule is $\dfrac{108}{100r}$ or $\dfrac{108}{i}$.

(We choose 108 instead of 110 because 108 has more factors.)

47. False. The correct solution is $|y| = e^{kx+C}$, which can be written (with a new C) as $y = Ce^{kx}$.

48. True. The differential equation is solved by an exponential equation that can be written in any base. Note that $Ce^{2t} = C\left(3^{kt}\right)$ when $k = 2/(\ln 3)$.

49. D. $A(t) = A_0 e^{rt}$

$2 = 1e^{7r}$

$r = \dfrac{\ln(2)}{7} = 0.099$

$t = \dfrac{\ln(3)}{0.099} = 11.1$

50. C. $A = A_0\left(\dfrac{1}{2}\right)^{t/r}$

$1 = 100\left(\dfrac{1}{2}\right)^{199/r}$

$\ln(.01) = \dfrac{199}{r}\ln(.5)$

$r = \dfrac{199\ \ln(.5)}{\ln(.01)} = 30$

51. D.

52. E. $T - 68 = (425 - 68)e^{-kt}$

$195 - 68 = 357e^{-30k}$

$e^{-30k} = \dfrac{127}{357} = 0.356$

$k = \dfrac{\ln(0.356)}{-30} = .0344$

$100 = 68 + 357e^{(-0.0344)t}$

$t = \dfrac{\ln\left(\dfrac{100-68}{357}\right)}{-0.0344} = 70\text{ min}$

$70 - 30 = 40$

53. (a) Since acceleration is $\dfrac{dv}{dt}$, we have Force $= m\dfrac{dv}{dt} = -kv$.

(b) From $m\dfrac{dv}{dt} = -kv$ we get $\dfrac{dv}{dt} = -\dfrac{k}{m}v$, which is the differential equation for exponential growth modeled by $v = Ce^{-(k/m)t}$. Since $v = v_0$ at $t = 0$, it follows that $C = v_0$.

(c) In each case, we would solve $2 = e^{-(k/m)t}$. If k is constant, an increase in m would require an increase in t. The object of larger mass takes longer to slow down. Alternatively, one can consider the equation $\dfrac{dv}{dt} = -\dfrac{k}{m}v$ to see that v changes more slowly for larger values of m.

54. (a) $s(t) = \int v_0 e^{-(k/m)t}\,dt = -\dfrac{v_0 m}{k}e^{-(k/m)t} + C$

Initial condition: $s(0) = 0$

$0 = -\dfrac{v_0 m}{k} + C$

$\dfrac{v_0 m}{k} = C$

$s(t) = -\dfrac{v_0 m}{k}e^{-(k/m)t} + \dfrac{v_0 m}{k}$

$= \dfrac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right)$

(b) $\lim\limits_{t\to\infty} s(t) = \lim\limits_{t\to\infty} \dfrac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right) = \dfrac{v_0 m}{k}$

55. $\dfrac{v_0 m}{k} = \text{coasting distance}$

$\dfrac{(0.80)(49.90)}{k} = 1.32$

$k = \dfrac{998}{33}$

We know that $\dfrac{v_0 m}{k} = 1.32$ and $\dfrac{k}{m} = \dfrac{998}{33(49.9)} = \dfrac{20}{33}$.

We have:

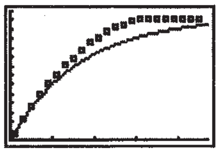
$s(t) = \dfrac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right)$

$= 1.32\left(1 - e^{-20t/33}\right)$

$\approx 1.32\left(1 - e^{-0.606t}\right)$

55. Continued

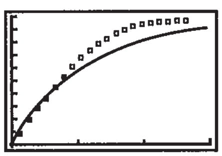
A graph of the model is shown superimposed on a graph of the data.



[0, 4.7] by [0, 1.4]

56.
$$\frac{v_0 m}{k} = \text{coasting distance}$$
$$\frac{(0.86)(30.84)}{k} = 0.97$$
$$k \approx 27.343$$
$$s(t) = \frac{v_0 m}{k}(1 - e^{-(k/m)t})$$
$$s(t) = 0.97(1 - e^{-(27.343/30.84)t})$$
$$s(t) = 0.97(1 - e^{-0.8866t})$$

A graph of the model is shown superimposed on a graph of the data.



[0, 3] by [0, 1]

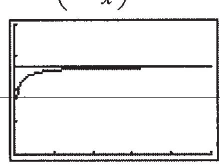
57. (a)

x	$\left(1+\frac{1}{x}\right)^x$
10	2.5937
100	2.7048
1000	2.7169
10,000	2.7181
100,000	2.7183

$e \approx 2.7183$

Graphical support:

$$y_1 = \left(1 + \frac{1}{x}\right)^x,\ y_2 = e$$



[0, 50] by [0, 4]

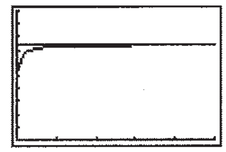
(b) $r = 2$

x	$\left(1+\frac{2}{x}\right)^x$
10	6.1917
100	7.2446
1000	7.3743
10,000	7.3876
100,000	7.3889

$e^2 \approx 7.389$

Graphical support:

$$y_1 = \left(1 + \frac{2}{x}\right)^x,\ y_2 = e^2$$



[0, 500] by [0, 10]

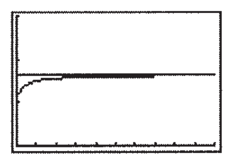
$r = 0.5$

x	$\left(1+\frac{0.5}{x}\right)^x$
10	1.6289
100	1.6467
1000	1.6485
10,000	1.6487
100,000	1.6487

$e^{0.5} \approx 1.6487$

Graphical support:

$$y_1 = \left(1 + \frac{0.5}{x}\right)^x,\ y_2 = e^{0.5}$$



[0, 10] by [0, 3]

(c) As we compound more times, the increment of time between compounding approaches 0. Continuous compounding is based on an instantaneous rate of change which is a limit of average rates as the increment in time approaches 0.

58. (a) To simplify calculations somewhat, we may write:

$$v(t) = \sqrt{\frac{mg}{k}}\frac{(e^{at} - e^{-at})e^{at}}{(e^{at} + e^{-at})e^{at}}$$
$$= \sqrt{\frac{mg}{k}}\frac{e^{2at} - 1}{e^{2at} + 1}$$
$$= \sqrt{\frac{mg}{k}}\frac{(e^{2at} + 1) - 2}{e^{2at} + 1}$$
$$= \sqrt{\frac{mg}{k}}\left(1 - \frac{2}{e^{2at} + 1}\right)$$

The left side of the differential equation is:

$$m\frac{dv}{dt} = m\sqrt{\frac{mg}{k}}(2)(e^{2at} + 1)^{-2}(2ae^{2at})$$
$$= 4ma\sqrt{\frac{mg}{k}}(e^{2at} + 1)^{-2}(e^{2at})$$
$$= 4m\sqrt{\frac{gk}{m}}\sqrt{\frac{mg}{k}}(e^{2at} + 1)^{-2}(e^{2at})$$
$$= \frac{4mge^{2at}}{(e^{2at} + 1)^2}$$

58. Continued

The right side of the differential equation is:

$$\begin{aligned} mg - kv^2 &= mg - k\left(\frac{mg}{k}\right)\left(1-\frac{2}{e^{2at}+1}\right)^2 \\ &= mg\left[1-\left(1-\frac{2}{e^{2at}+1}\right)^2\right] \\ &= mg\left(1-1+\frac{4}{e^{2at}+1}-\frac{4}{(e^{2at}+1)^2}\right) \\ &= mg\frac{4(e^{2at}+1)-4}{(e^{2at}+1)^2} = \frac{4\,mg\,e^{2at}}{(e^{2at}+1)^2} \end{aligned}$$

Since the left and right sides are equal, the differential equation is satisfied.

And $v(0) = \sqrt{\frac{mg}{k}}\frac{e^0-e^0}{e^0+e^0} = 0$, so the initial condition is also satisfied.

(b) $$\begin{aligned} \lim_{t\to\infty} v(t) &= \lim_{t\to\infty}\left(\sqrt{\frac{mg}{k}}\frac{e^{at}-e^{-at}}{e^{at}+e^{-at}}\cdot\frac{e^{-at}}{e^{-at}}\right) \\ &= \lim_{t\to\infty}\left(\sqrt{\frac{mg}{k}}\frac{1-e^{-2at}}{1+e^{-2at}}\right) \\ &= \sqrt{\frac{mg}{k}}\left(\frac{1-0}{1+0}\right) = \sqrt{\frac{mg}{k}} \end{aligned}$$

The limiting velocity is $\sqrt{\frac{mg}{k}}$.

(c) $\sqrt{\frac{mg}{k}} = \sqrt{\frac{160}{0.005}} \approx 179$ ft/sec

The limiting velocity is about 179 ft/sec, or about 122 mi/hr.

Section 6.5 Logistic Growth (pp. 362–376)

Exploration 1 Exponential Growth Revisited

1. $100(2)^{12} = 409600$

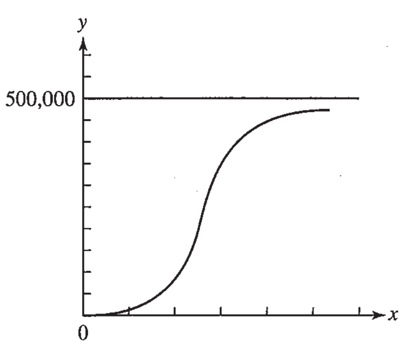
2. $100(2)^{(12\cdot 24)} = 4.97\times 10^{88}$

3. No. This number is much larger than the estimated number of atoms.

4. $500{,}000 = 100(2)^x$

   $\frac{\log 5000}{\log 2} = x = 12.29$ hours

5.



Exploration 2 Learning From the Differential Equation

1. $\frac{dP}{dt}$ will be close to zero when P is close to 0 and when P is close to M.

2. P is half the value of M at its vertex.

3. When $P = \frac{M}{2}$, $\frac{dP}{dt} = kP(M-P)$ is at its maximum.

4. When the initial population is less than M, the initial growth rate is positive.

5. When the initial population is more than M, the initial growth rate is negative.

6. When the initial population is equal to M, the growth rate is 0.

7. $\lim_{t\to\infty} P(t) = M$, regardless of the initial population. The limit depends only on M.

Quick Review 6.5

1. $$\begin{array}{r} x+1 \\ x-1\overline{)\,x^2} \\ \underline{x^2-x} \\ x \\ \underline{x-1} \\ 1 \end{array}$$

   $x+1+\frac{1}{x-1}$

2. $$\begin{array}{r} 1 \\ x^2-4\overline{)\,x^2} \\ \underline{x^2-4} \\ 4 \end{array}$$

   $1+\frac{4}{x^2-4}$

3. $$\begin{array}{r} 1 \\ x^2+x-2\overline{)\,x^2+x+1} \\ \underline{x^2+x-2} \\ 3 \end{array}$$

   $1+\frac{3}{x^2+x-2}$

4. $$\begin{array}{r} x \\ x^2-1\overline{)\,x^3-5} \\ \underline{x^3-x} \\ x-5 \end{array}$$

   $x+\frac{x-5}{x^3-5}$

5. $(-\infty, \infty)$

6. $\lim_{x\to\infty}\frac{60}{1+5e^{-0.1x}} = \frac{60}{1+5(0)} = 60$