84. (a) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{1+x^2}} = \int_{\tan^{-1}0}^{\tan^{-1}\sqrt{3}} \frac{\sec^2 u\,du}{\sqrt{1+\tan^2 u}}$

$= \int_0^{\pi/3} \frac{\sec^2 u\,du}{\sec u} = \int_0^{\pi/3} \sec u\,du$

(b) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{1+x^2}} = \int_0^{\pi/3} \sec u\,du = \left[\ln|\sec u + \tan u|\right]_0^{\pi/3}$

$= \ln(2+\sqrt{3}) - \ln(1+0) = \ln(2+\sqrt{3})$

Section 6.3 Antidifferentiation by Parts (pp. 341–349)

Exploration 1 Choosing the Right *u* and *dv*

1. $u = 1 \quad du = 0$

$dv = x\cos x \quad v = \int x\cos x\,dx$

Using 1 for u is never a good idea because it places us back where we started.

2. $u = x\cos x \quad du = \cos x - x\sin x$

$dv = dx \quad v = \int dx = 1$

The selection of $u = x\cos x$ will place a more difficult integral into $\int v\,du.$

3. $u = \cos x \quad du = -\sin x$

$dv = x\,dx \quad v = \int x\,dx = x^2$

The selection of $dv = x\,dx$ will place a more difficult integral into $\int v\,du.$

4. $u = x$ and $dv = \cos x\,dx$ are good choices because the integral is simplified.

Quick Review 6.3

1. $\frac{dy}{dx} = (x^3)(\cos 2x)(2) + (\sin 2x)(3x^2)$

$= 2x^3\cos 2x + 3x^2\sin 2x$

2. $\frac{dy}{dx} = (e^{2x})\left(\frac{3}{3x+1}\right) + \ln(3x+1)(2e^{2x})$

$= \frac{3e^{2x}}{3x+1} + 2e^{2x}\ln(3x+1)$

3. $\frac{dy}{dx} = \frac{1}{1+(2x)^2} \bullet 2$

$= \frac{2}{1+4x^2}$

4. $\frac{dy}{dx} = \frac{1}{\sqrt{1-(x+3)^2}}$

5. $y = \tan^{-1} 3x$

$\tan y = 3x$

$x = \frac{1}{3}\tan y$

6. $y = \cos^{-1}(x+1)$

$\cos y = x+1$

$x = \cos y - 1$

7. $\int_0^1 \sin \pi x\,dx = -\frac{1}{\pi}\cos \pi x\Big]_0^1$

$= -\frac{1}{\pi}\cos \pi + \frac{1}{\pi}\cos 0$

$= -\frac{1}{\pi}(-1) + \frac{1}{\pi} = \frac{2}{\pi}$

8. $\frac{dy}{dx} = e^{2x}$

$dy = e^{2x}dx$

Integrate both sides.

$\int dy = \int e^{2x}dx$

$y = \frac{1}{2}e^{2x} + C$

9. $\frac{dy}{dx} = x + \sin x$

$dy = (x+\sin x)dx$

Integrate both sides.

$\int dy = \int (x+\sin x)dx$

$y = \frac{1}{2}x^2 - \cos x + C$

$y(0) = -1 + C = 2$

$C = 3$

$y = \frac{1}{2}x^2 - \cos x + 3$

10. $\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x - \cos x)\right)$

$= \frac{1}{2}e^x(\cos x + \sin x) + (\sin x - \cos x)\frac{1}{2}e^x$

$= \frac{1}{2}e^x\cos x + \frac{1}{2}e^x\sin x + \frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x$

$= e^x\sin x$

Section 6.3 Exercises

1. $\int x\sin x\,dx$

$dv = \sin x\,dx \quad v = \int \sin x\,dx = -\cos x$

$u = x \quad du = dx$

$-x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$

2. $\int xe^x dx$

$dv = e^x dx \quad v = \int e^x dx = e^x$

$u = x \quad du = dx$

$xe^x - \int e^x dx = xe^x - e^x + C$

3. $\int 3t\,e^{2t}dt$

$dv = e^{2t}dt \qquad v = \int e^{2t}dt = \frac{e^{2t}}{2}$

$u = 3t \qquad du = 3\,dt$

$3t\frac{e^{2t}}{2} - \int 3\frac{e^{2t}}{2}dt = \frac{3}{2}te^{2t} - \frac{3}{4}e^{2t} + C$

4. $\int 2t\cos(3t)\,dt$

$dv = \cos 3t\,dt \quad v = \int \cos(3t)\,dt = \frac{\sin 3t}{3}$

$u = 2t \qquad du = 2\,dt$

$2t\frac{\sin 3t}{3} - \int 2\frac{\sin 3t}{3}dt = \frac{2}{3}t\sin 3t - \frac{2}{9}\cos(3t) + C$

5. $\int x^2\cos x\,dx$

$dv = \cos x\,dx \qquad v = \int \cos x\,dx = \sin x$

$u = x^2 \qquad du = 2x\,dx$

$x^2\sin x - \int 2x\sin x\,dx$

$dv = \sin x\,dx \qquad v = \int \sin x\,dx = -\cos x$

$u = 2x \qquad du = 2dx$

$x^2\sin x + 2x\cos x - \int 2\cos x dx$

$= x^2\sin x + 2x\cos x - 2\sin x + C$

6. $\int x^2e^{-x}dx$

$dv = e^{-x}dx \qquad v = \int e^{-x}dx = -e^{-x}$

$u = x^2 \qquad du = 2x\,dx$

$-x^2e^{-x} - \int -2x\,e^{-x}dx = -x^2e^{-x} + \int 2xe^{-x} + C$

$dv = e^{-x} \qquad v = \int e^{-x}dx = -e^{-x}$

$u = 2x \qquad du = 2\,dx$

$-x^2e^{-x} - 2xe^{-x} - \int -2e^{-x}dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

7. $\int 3x^2\,e^{2x}dx$

$dv = e^{2x}dx \qquad v = \int e^{2x}dx = \frac{e^{2x}}{2}$

$u = 3x^2 \qquad du = 6x$

$3x^2\frac{e^{2x}}{2} - \int 6x\frac{e^{2x}}{2}dx = \frac{3}{2}x^2e^{2x} - \int 3x\,e^{2x}dx$

$dv = e^{2x} \qquad v = \int e^{2x}dx = \frac{e^{2x}}{2}$

$u = 3x \qquad du = 3dx$

$\frac{3}{2}x^2e^{2x} - \left[\frac{3}{2}xe^{2x} - \int 3\frac{e^{2x}}{2}dx\right] = \frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C$

8. $\int x^2\cos\left(\frac{x}{2}\right)dx$

$dv = \cos\left(\frac{x}{2}\right)dx \qquad v = \int \cos\left(\frac{x}{2}\right)dx = 2\sin\left(\frac{x}{2}\right)$

$u = x^2 \qquad du = 2x\,dx$

$2x^2\sin\left(\frac{x}{2}\right) - \int 4x\sin\left(\frac{x}{2}\right)dx$

$dv = \sin\left(\frac{x}{2}\right) \qquad v = \int \sin\left(\frac{x}{2}\right)dx = -2\cos\left(\frac{x}{2}\right)$

$u = 4x \qquad du = 4dx$

$2x^2\sin\left(\frac{x}{2}\right) + 8x\cos\left(\frac{x}{2}\right) - \int 8\cos\left(\frac{x}{2}\right)dx$

$= 2x^2\sin\left(\frac{x}{2}\right) + 8x\cos\left(\frac{x}{2}\right) - 16\sin\left(\frac{x}{2}\right) + C$

9. $\int y\ln y\,dy$

$dv = y\,dy \qquad v = \int y\,dy = \frac{y^2}{2}$

$u = \ln y \quad du = \frac{1}{y}dy$

$\frac{1}{2}y^2\ln y - \int \frac{y^2}{2}\frac{1}{y}dy = \frac{1}{2}y^2\ln y - \frac{y^2}{4} + C$

10. $\int t^2\ln t\,dt$

$dv = t^2\,dt \qquad v = \int t^2dt = \frac{t^3}{3}$

$u = \ln t \qquad du = \frac{1}{t}dt$

$\frac{1}{3}t^3\ln t - \int \frac{t^3}{3}\frac{1}{t}dt = \frac{1}{3}t^3\ln t - \frac{t^3}{9} + C$

11. $\int dy = \int ((x+2)\sin x)dx$

$dv = \sin x\,dx \qquad v = \int \sin x\,dx = -\cos x$

$u = x+2 \qquad du = dx$

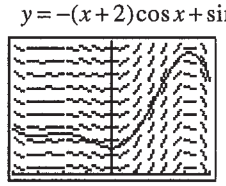
$-(x+2)\cos x - \int(-\cos x)\,dx = -(x+2)\cos x + \sin x + C$

$2 = -(0+2)\cos(0) + \sin(0) + C$

$2 = -2 + C$

$C = 4$

$y = -(x+2)\cos x + \sin x + 4$



[–4, 4] by [0, 10]

12. $\int dy = \int 2xe^{-x}dx$

$dv = e^{-x}dx \qquad v = \int e^{-x}dx = -e^{-x}$

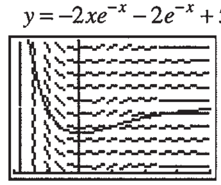
$u = 2x \qquad du = 2\,dx$

$-2xe^{-x} - \int -2e^{-x}dx = -2xe^{-x} - 2e^{-x} + C$

$3 = -2(0)\,e^{(-0)} - 2e^{(-0)} + C$

$5 = C$

$y = -2xe^{-x} - 2e^{-x} + 5$



[–2, 4] by [0, 10]

13. $\int du = \int x \sec^2 x\,dx$

$dv = \sec^2 x\,dx \quad v = \int \sec^2 x\,dx = \tan x$

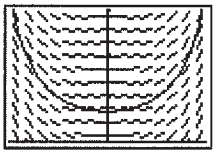
$w = x \quad dw = dx$

$x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$

$1 = 0\tan(0) + \ln|\cos(0)| + C$

$C = 1$

$u = x\tan(x) + \ln|\cos(x)| + 1$



[−1.2, 1.2] by [0, 3]

14. $\int dz = x^3 \ln x\,dx$

$dv = x^3 dx \quad v = \int x^3 dx = \frac{x^4}{4}$

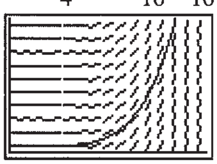
$u = \ln x \quad du = \frac{1}{x}dx$

$\frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{1}{x}dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

$5 = \frac{(1)^4}{4}\ln(1) - \frac{(1)^4}{16} + C$

$C = \frac{81}{16}$

$z = \frac{x^4}{4}\ln x - \frac{x^4}{16} + \frac{81}{16}$



[0, 5] by [0, 100]

15. $\int dy = \int x\sqrt{x-1}\,dx$

$dv = (x-1)^{1/2}dx \quad v = \int (x-1)^{1/2}dx = \frac{2}{3}(x-1)^{3/2}$

$u = x \quad du = dx$

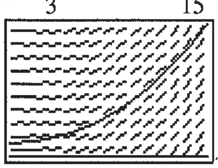
$\frac{2}{3}x(x-1)^{3/2} - \int \frac{2}{3}(x-1)^{3/2}dx$

$= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C$

$2 = \frac{2}{3}(1)(1-1)^{3/2} - \frac{4}{15}(1-1)^{5/2} + C$

$C = 2$

$y = \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + 2$



[1, 5] by [0, 20]

16. $\int dy = \int 2x\sqrt{x+2}\,dx$

$dv = (x+2)^{1/2}dx \quad v = \int (x+2)^{1/2}dx = \frac{2}{3}(x+2)^{3/2}$

$u = 2x \quad du = 2dx$

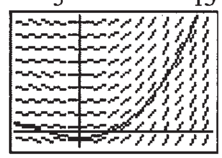
$\frac{4}{3}x(x+2)^{3/2} - \int \frac{4}{3}(x+2)^{3/2}dx$

$= \frac{4}{3}x(x+2)^{3/2} - \frac{8}{15}(x+2)^{5/2} + C$

$0 = \frac{4}{3}(-1)(-1+2)^{3/2} - \frac{8}{15}(-1+2)^{5/2} + C$

$C = \frac{28}{15}$

$y = \frac{4}{3}x(x+2)^{3/2} - \frac{8}{15}(x+2)^{5/2} + \frac{28}{15}$



[−2, 4] by [−3, 25]

17. $\int e^x \sin x\,dx$

$dv = e^x\,dx \quad v = \int e^x\,dx = e^x$

$u = \sin x \quad du = \cos x\,dx$

$e^x \sin x - \int e^x \cos x\,dx$

$dv = e^x\,dx \quad v = \int e^x\,dx = e^x$

$u = \cos x \quad du = -\sin x\,dx$

$\int e^x \sin x\,dx = e^x \sin x - \left[e^x \cos x - \int -e^x \sin x\,dx\right]$

$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$

18. $\int e^{-x}\cos x\,dx$

$dv = \cos x\,dx \quad v = \int \cos x\,dx = \sin x$

$u = e^{-x} \quad du = -e^{-x}\,dx$

$e^{-x}\sin x - \int -e^{-x}\sin x\,dx$

$dv = \sin x\,dx \quad v = \int \sin x\,dx = -\cos x$

$u = e^{-x} \quad du = -e^{-x}\,dx$

$\int e^{-x}\cos x\,dx = e^{-x}\sin x - \left[e^{-x}\cos x - \int -e^{-x}\cos x\,dx\right]$

$\int e^{-x}\cos x\,dx = \frac{e^{-x}}{2}(\sin x - \cos x) + C$

19. $\int e^x \cos 2x\, dx$

$dv = \cos 2x\, dx \quad v = \int \cos 2x\, dx = \frac{1}{2}\sin 2x$

$u = e^x \qquad du = e^x dx$

$\frac{1}{2}e^x \sin 2x - \int \frac{1}{2}\sin 2x\, e^x dx$

$dv = \frac{1}{2}\sin 2x dx \qquad v = \int \frac{1}{2}\sin 2x\, dx = -\frac{1}{4}\cos 2x$

$u = e^x \qquad du = e^x dx$

$\int e^x \cos 2x\, dx$

$= \frac{1}{2}e^x \sin 2x - \left[-\frac{1}{4}e^x \cos 2x - \int -\frac{1}{4}\cos 2x\, e^x dx\right]$

$\int e^x \cos 2x\, dx = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x dx$

$\frac{5}{4}\int e^x \cos 2x dx = \frac{1}{4}e^x(2\sin 2x + \cos 2x)$

$\int e^x \cos 2x dx = \frac{e^x}{5}(2\sin 2x + \cos 2x) + C$

20. $\int e^{-x}\sin 2x\, dx$

$dv = \sin 2x\, dx \quad v = \int \sin 2x\, dx = -\frac{1}{2}\cos 2x$

$u = e^{-x} \qquad du = -e^{-x}dx$

$-\frac{1}{2}e^{-x}\cos 2x - \int \frac{1}{2}\cos 2x\, e^{-x}dx$

$dv = \frac{1}{2}\cos 2x\, dx \quad v = \int \frac{1}{2}\cos 2x\, dx = \frac{1}{4}\sin 2x$

$u = e^{-x} \qquad du = -e^{-x}dx$

$\int e^{-x}\sin 2x\, dx$

$= -\frac{1}{2}e^{-x}\cos 2x - \left[\frac{1}{4}e^{-x}\sin 2x - \int -\frac{1}{4}e^{-x}\sin 2x\, dx\right]$

$\int e^{-x}\sin 2x\, dx$

$= -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}\int e^{-x}\sin 2x\, dx$

$\frac{5}{4}\int e^{-x}\sin 2x\, dx = -\frac{1}{4}e^{-x}(2\cos 2x + \sin 2x)$

$\int e^{-x}\sin 2x\, dx = -\frac{e^{-x}}{5}(2\cos 2x + \sin 2x) + C$

21. Use tabular integration with $f(x) = x^4$ and $g(x) = e^{-x}$.

f(x) and its derivatives		g(x) and its integrals
x^4	(+)	e^{-x}
$4x^3$	(−)	$-e^{-x}$
$12x^2$	(+)	e^{-x}
$24x$	(−)	$-e^{-x}$
24	(+)	e^{-x}
0		$-e^{-x}$

$\int x^4 e^{-x}\, dx$

$= -x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x} + C$

$= -(x^4 + 4x^3 + 12x^2 + 24x + 24)e^{-x} + C$

22. Use tabular integration with $f(x) = x^2 - 5x$ and $g(x) = e^x$

f(x) and its derivatives		g(x) and its integrals
$x^2 - 5x$	(+)	e^x
$2x - 5$	(−)	e^x
2	(+)	e^x
0		e^x

$\int (x^2 - 5x)e^x\, dx = (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x$

$= (x^2 - 7x + 7)e^x$

23. Use tabular integration with $f(x) = x^3$ and $g(x) = e^{-2x}$.

f(x) and its derivatives		g(x) and its integrals
x^3	(+)	e^{-2x}
$3x^2$	(−)	$-\frac{1}{2}e^{-2x}$
$6x$	(+)	$\frac{1}{4}e^{-2x}$
6	(−)	$-\frac{1}{8}e^{-2x}$
0		$\frac{1}{16}e^{-2x}$

$\int x^3 e^{-2x}\, dx$

$= -\frac{1}{2}x^3e^{-2x} - \frac{3}{4}x^2e^{-2x} - \frac{3}{4}xe^{-2x} - \frac{3}{8}e^{-2x} + C$

$= -\left(\frac{x^3}{2} + \frac{3x^2}{4} + \frac{3x}{4} + \frac{3}{8}\right)e^{-2x} + C$

24. Use tabular integration with $f(x) = x^3$ and $g(x) = \cos 2x$.

f(x) and its derivatives		g(x) and its integrals
x^3	(+)	$\cos 2x$
$3x^2$	(−)	$\frac{1}{2}\sin 2x$
$6x$	(+)	$-\frac{1}{4}\cos 2x$
6	(−)	$-\frac{1}{8}\sin 2x$
0		$\frac{1}{16}\cos 2x$

$\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x + C$

25. Use tabular integration with $f(x) = x^2$ and $g(x) = \sin 2x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	(+)	$\sin 2x$
$2x$	(−)	$-\frac{1}{2}\cos 2x$
2	(+)	$-\frac{1}{4}\sin 2x$
0		$\frac{1}{8}\cos 2x$

$$\int x^2 \sin 2x\, dx = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$
$$= \left(\frac{1-2x^2}{4}\right)\cos 2x + \frac{x}{2}\sin 2x + C$$
$$\int_0^{\pi/2} x^2 \sin 2x\, dx = \left[\left(\frac{1-2x^2}{4}\right)\cos 2x + \frac{x}{2}\sin 2x\right]_0^{\pi/2}$$
$$= \left(\frac{1-2\left(\frac{\pi}{2}\right)^2}{4}\right)(-1) + 0 - \left(\frac{1}{4}\right)(1) - 0$$
$$= \frac{\pi^2}{8} - \frac{1}{2} \approx 0.734$$

Check: $\text{NINT}\left(x^2\sin 2x,\, x,\, 0,\, \frac{\pi}{2}\right) \approx 0.734$

26. Use tabular integration with $f(x) = x^3$ and $g(x) = \cos 2x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\cos 2x$
$3x^2$	(−)	$\frac{1}{2}\sin 2x$
$6x$	(+)	$-\frac{1}{4}\cos 2x$
6	(−)	$-\frac{1}{8}\sin 2x$
0		$\frac{1}{16}\cos 2x$

$$\int x^3\cos 2x\, dx = \frac{1}{2}x^3\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x$$
$$\sin 2x - \frac{3}{8}\cos 2x$$
$$= \left(\frac{x^3}{2} - \frac{3x}{4}\right)\sin 2x + \left(\frac{3x^2}{4} - \frac{3}{8}\right)\cos 2x + C$$

$$\int_0^{\pi/2} x^3\cos 2x\, dx = \left[\left(\frac{x^3}{2} - \frac{3x}{4}\right)\sin 2x + \left(\frac{3x^2}{4} - \frac{3}{8}\right)\cos 2x\right]_0^{\pi/2}$$
$$= 0 + \left(\frac{3\pi^2}{16} - \frac{3}{8}\right)(-1) - 0 - \left(-\frac{3}{8}\right)(1)$$
$$= \frac{3}{4} - \frac{3\pi^2}{16} \approx -1.101$$

Check: $\text{NINT}\left(x^3\cos 2x,\, x,\, 0,\, \frac{\pi}{2}\right) \approx -1.101$

27. Let $u = e^{2x}$ $\quad dv = \cos 3x\, dx$

$du = 2e^{2x}\, dx \quad v = \frac{1}{3}\sin 3x$

$$\int e^{2x}\cos 3x\, dx = (e^{2x})\left(\frac{1}{3}\sin 3x\right) - \int\left(\frac{1}{3}\sin 3x\right)(2e^{2x}\, dx)$$
$$= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\, dx$$

Let $u = e^{2x}$ $\quad dv = \sin 3x\, dx$

$du = 2e^{2x}\, dx \quad v = -\frac{1}{3}\cos 3x$

$$\int e^{2x}\cos 3x\, dx = \frac{1}{3}e^{2x}\sin 3x$$
$$-\frac{2}{3}\left[(e^{2x})\left(-\frac{1}{3}\cos 3x\right) - \int\left(-\frac{1}{3}\cos 3x\right)(2e^{2x}\, dx)\right]$$
$$= \frac{1}{9}e^{2x}(3\sin 3x + 2\cos 3x) - \frac{4}{9}\int e^{2x}\cos 3x\, dx$$
$$\frac{13}{9}\int e^{2x}\cos 3x\, dx = \frac{1}{9}e^{2x}(3\sin 3x + 2\cos 3x)$$
$$\int e^{2x}\cos 3x\, dx = \frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x)$$
$$\int_{-2}^{3} e^{2x}\cos 3x\, dx = \left[\frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x)\right]_{-2}^{3}$$
$$= \frac{1}{13}[e^6(3\sin 9 + 2\cos 9)$$
$$-e^{-4}(3\sin(-6) + 2\cos(-6)]$$
$$= \frac{1}{13}[e^6(2\cos 9 + 3\sin 9)$$
$$-e^{-4}(2\cos 6 - 3\sin 6)]$$
$$\approx -18.186$$

Check: $\text{NINT}\left(e^{2x}\cos 3x,\, x, -2,\, 3\right) \approx -18.186$

28. Let $u = e^{-2x}$ $\quad dv = \sin 2x\, dx$

$du = -2e^{-2x}\, dx \quad v = -\frac{1}{2}\cos 2x$

$$\int e^{-2x}\sin 2x\, dx = (e^{-2x})\left(-\frac{1}{2}\cos 2x\right)$$
$$-\int\left(-\frac{1}{2}\cos 2x\right)(-2e^{-2x}\, dx)$$
$$= -\frac{1}{2}e^{-2x}\cos 2x - \int e^{-2x}\cos 2x\, dx$$

28. Continued

Let $u=e^{-2x}$ $\quad dv=\cos 2x\,dx$

$du=-2e^{-2x}dx$ $\quad v=\frac{1}{2}\sin 2x$

$$\int e^{-2x}\sin 2x\,dx=-\frac{1}{2}e^{-2x}\cos 2x-\left[(e^{-2x})\left(\frac{1}{2}\sin 2x\right)-\int\left(\frac{1}{2}\sin 2x\right)(-2e^{-2x}\,dx)\right]$$

$$=-\frac{1}{2}e^{-2x}(\cos 2x+\sin 2x)-\int e^{-2x}\sin 2x\,dx$$

$$2\int e^{-2x}\sin 2x\,dx=-\frac{1}{2}e^{-2x}(\cos 2x+\sin 2x)+C$$

$$\int e^{-2x}\sin 2x\,dx=-\frac{e^{-2x}}{4}(\cos 2x+\sin 2x)+C$$

$$\int_{-3}^{2}e^{-2x}\sin 2x\,dx=\left[-\frac{e^{-2x}}{4}(\cos 2x+\sin 2x)\right]_{-3}^{2}$$

$$=-\frac{e^{-4}}{4}(\cos 4+\sin 4)+\frac{e^{6}}{4}[\cos(-6)+\sin(-6)]$$

$$=-\frac{e^{-4}}{4}(\cos 4+\sin 4)+\frac{e^{6}}{4}(\cos 6-\sin 6)$$

$$\approx 125.028$$

Check: $\text{NINT}\left(e^{-2x}\sin 2x,\,x,\,-3,\,2\right)\approx 125.028$

29. $y=\int x^2e^{4x}\,dx$

Let $u=x^2$ $\quad dv=e^{4x}dx$

$du=2x\,dx$ $\quad v=\frac{1}{4}e^{4x}$

$$y=(x^2)\left(\frac{1}{4}e^{4x}\right)-\int\left(\frac{1}{4}e^{4x}\right)(2x\,dx)$$

$$=\frac{1}{4}x^2e^{4x}-\frac{1}{2}\int xe^{4x}dx$$

Let $u=x$ $\quad dv=e^{4x}\,dx$

$du=dx$ $\quad v=\frac{1}{4}e^{4x}$

$$y=\frac{1}{4}x^2e^{4x}-\frac{1}{2}\left[(x)\left(\frac{1}{4}e^{4x}\right)-\int\left(\frac{1}{4}e^{4x}\right)dx\right]$$

$$y=\frac{1}{4}x^2e^{4x}-\frac{1}{8}xe^{4x}+\frac{1}{32}e^{4x}+C$$

$$y=\left(\frac{x^2}{4}-\frac{x}{8}+\frac{1}{32}\right)e^{4x}+C$$

30. $y=\int x^2\ln x\,dx$

Let $u=\ln x$ $\quad dv=x^2\,dx$

$du=\frac{1}{x}dx$ $\quad v=\frac{1}{3}x^3$

$$y=(\ln x)\left(\frac{1}{3}x^3\right)-\int\left(\frac{1}{3}x^3\right)\left(\frac{1}{x}dx\right)$$

$$y=\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2dx$$

$$y=\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$$

31. $y=\int\theta\sec^{-1}\theta\,d\theta$

Let $u=\sec^{-1}\theta$ $\quad dv=\theta\,d\theta$

$du=\frac{1}{\theta\sqrt{\theta^2-1}}du$ $\quad v=\frac{1}{2}\theta^2$

Note that we are told $\theta>1$, so no absolute value is needed in the expression for du.

$$y=(\sec^{-1}\theta)\left(\frac{1}{2}\theta^2\right)-\int\left(\frac{1}{2}\theta^2\right)\left(\frac{1}{\theta\sqrt{\theta^2-1}}d\theta\right)$$

$$y=\frac{\theta^2}{2}\sec^{-1}\theta-\frac{1}{2}\int\frac{\theta\,d\theta}{\sqrt{\theta^2-1}}$$

Let $w=\theta^2-1,\ dw=2\theta\,d\theta$

$$y=\frac{\theta^2}{2}\sec^{-1}\theta-\frac{1}{4}\int w^{-1/2}dw$$

$$y=\frac{\theta^2}{2}\sec^{-1}\theta-\frac{1}{2}w^{1/2}+C$$

$$y=\frac{\theta^2}{2}\sec^{-1}\theta-\frac{1}{2}\sqrt{\theta^2-1}+C$$

32. $y=\int\theta\sec\theta\tan\theta\,d\theta$

Let $u=\theta$ $\quad dv=\sec\theta\tan\theta\,d\theta$

$du=d\theta$ $\quad v=\sec\theta$

$$y=\theta\sec\theta-\int\sec\theta\,d\theta$$

$$y=\theta\sec\theta-\ln|\sec\theta+\tan\theta|+C$$

Note : In the last step, we used the result of Exercise 45 in Section 6.2.

33. Let $u=x$ $\quad dv=\sin x\,dx$

$du=dx$ $\quad v=-\cos x$

$$\int x\sin x\,dx=-x\cos x+\int\cos x\,dx$$

$$=-x\cos x+\sin x+C$$

(a) $\int_0^{\pi}|x\sin x|\,dx=\int_0^{\pi}x\sin x\,dx$

$$=\left[-x\cos x+\sin x\right]_0^{\pi}$$

$$=-\pi(-1)+0+0(1)-0$$

$$=\pi$$

(b) $\int_{\pi}^{2\pi}|x\sin x|\,dx=-\int_{\pi}^{2\pi}x\sin x\,dx$

$$=\left[x\cos x-\sin x\right]_{\pi}^{2\pi}$$

$$=2\pi(1)-0-\pi(-1)+0$$

$$=3\pi$$

(c) $\int_0^{2\pi}|x\sin x|\,dx=\int_0^{\pi}|x\sin x|\,dx+\int_{\pi}^{2\pi}|x\sin x|\,dx$

$$=\pi+3\pi=4\pi$$

34. We begin by evaluating $\int (x^2+x+1)e^{-x}\,dx$.

Let $u = x^2+x+1$ $\quad dv = e^{-x}\,dx$

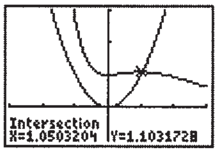
$du = (2x+1)\,dx$ $\quad v = -e^{-x}$

$$\int (x^2+x+1)e^{-x}\,dx$$
$$= -(x^2+x+1)e^{-x} + \int (2x+1)e^{-x}\,dx$$

Let $u = 2x+1$ $\quad dv = e^{-x}\,dx$

$du = 2\,dx$ $\quad v = -e^{-x}$

$$\int (x^2+x+1)e^{-x}\,dx$$
$$= -(x^2+x+1)e^{-x} - (2x+1)e^{-x} + \int 2e^{-x}\,dx$$
$$= -(x^2+x+1)e^{-x} - (2x+1)e^{-x} - 2e^{-x} + C$$
$$= -(x^2+3x+4)e^{-x} + C$$



$[-3, 3]$ by $[-3, 3]$

The graph shows that the two curves intersect at $x = k$, where $k \approx 1.050$. The area we seek is

$$\int_0^k (x^2+x+1)e^{-x} - \int_0^k x^2\,dx$$
$$= \left[-(x^2+3x+4)e^{-x}\right]_0^k - \left[\frac{1}{3}x^3\right]_0^k$$
$$\approx (-2.888+4) - (0.386-0)$$
$$\approx 0.726$$

35. First, we evaluate $\int e^{-t}\cos t\,dt$.

Let $u = e^{-t}$ $\quad dv = \cos t\,dt$

$du = -e^{-t}\,dt$ $\quad v = \sin t$

$$\int e^{-t}\cos t\,dt = e^{-t}\sin t + \int \sin t\, e^{-t}\,dt$$

Let $u = e^{-t}$ $\quad dv = \sin t\,dt$

$du = -e^{-t}\,dt$ $\quad v = -\cos t$

$$\int e^{-t}\cos t\,dt = e^{-t}\sin t - e^{-t}\cos t - \int e^{-t}\cos t\,dt$$
$$2\int e^{-t}\cos t\,dt = e^{-t}(\sin t - \cos t) + C$$
$$\int e^{-t}\cos t\,dt = \frac{1}{2}e^{-t}(\sin t - \cos t) + C$$

Now we find the average value of $y = 2e^{-t}\cos t$ for $0 \le t \le 2\pi$.

$$\text{Average value} = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt$$
$$= \frac{1}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$$
$$= \frac{1}{2\pi}e^{-t}(\sin t - \cos t)\Big]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[e^{-2\pi}(-1) - e^0(-1)\right]$$
$$= \frac{1-e^{-2\pi}}{2\pi} \approx 0.159$$

36. True. Use parts, letting $u = x$, $dv = g(x)dx$, and $v = f(x)$.

37. True. Use parts, letting $u = x^2$, $dv = g(x)dx$, and $v = f(x)$.

38. B. $\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C$

See problem 5.

$\int 2x\sin x\,dx = -2x\cos x + 2\sin x + C$

See problem 1.

$h(x) = x^2\sin x + C$

39. B. $\int x\sin(5x)\,dx$

$dv = \sin(5x)\,dx$ $\quad v = \int \sin(5x)\,dx = -\frac{1}{5}\cos 5x$

$u = x$ $\quad du = dx$

$$-\frac{1}{5}x\cos(5x) - \int -\frac{1}{5}\cos(5x)\,dx$$
$$= -\frac{1}{5}x\cos(5x) + \frac{1}{25}\sin(5x) + C$$

40. C. $\int x\csc^2 x\,dx$

$dv = \csc^2 x\,dx$ $\quad v = \int \csc^2 x\,dx = -\cot x$

$u = x$ $\quad du = dx$

$$-x\cot x - \int -\cot x\,dx = -x\cot x + \ln|\sin x| + C$$

41. C. $\int dy = \int 4x\ln x\,dx$

$dv = 4x\,dx$ $\quad v = \int 4x\,dx = 2x^2$

$u = \ln x$ $\quad du = \frac{1}{x}\,dx$

$$2x^2\ln x - \int 2x^2\frac{1}{x}\,dx = 2x^2\ln x - x^2 + C$$

42. **(a)** Let $u = x$ $\quad dv = e^x\,dx$

$du = dx$ $\quad v = e^x$

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$
$$= xe^x - e^x + C$$
$$= (x-1)e^x + C$$

(b) Using the result from part (a):

Let $u = x^2$ $\quad dv = e^x\,dx$

$du = 2x\,dx$ $\quad v = e^x$

$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx$$
$$= x^2e^x - 2(x-1)e^x + C$$
$$= (x^2-2x+2)e^x + C$$

(c) Using the result from part (b):

Let $u = x^3$ $\quad dv = e^x\,dx$

$du = 3x^2dx$ $\quad v = e^x$

$$\int x^3e^x\,dx = x^3e^x - \int 3x^2e^x\,dx$$
$$= x^3e^x - 3(x^2-2x+2)e^x + C$$
$$= (x^3-3x^2+6x-6)e^x + C$$

(d) $\left[x^n - \frac{d}{dx}x^n + \frac{d}{dx^2}x^n - \cdots + (-1)^n\frac{d^n}{dx^n}x^n\right]e^x + C$

or $\left[x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^{n-1}(n)!x + (-1)^n(n!)\right]e^x + C$

42. Continued

(e) Use mathematical induction or argue based on tabular integration.

Alternately, show that the derivative of the answer to part (d) is $x^n e^x$:

$$\frac{d}{dx}\Big[\big(x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^{n-1}(n!)x + (-1)^n n!\big)e^x + C\Big]$$
$$= [x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^{n-1}(n!)x + (-1)^n n!]e^x + e^x \frac{d}{dx}\Big[x^n - nx^{n-1} + n(n-1)n^{n-2} - \cdots + (-1)^{n-1}(n!)x + (-1)^n n!]$$
$$= [x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^{n-1} + (n!)x + (-1)^n n!]e^x + \big[nx^{n-1} - n(n-1)x^{n-2} + n(n-1)(n-2)x^{n-3} - \cdots + (-1)^{n-1} n!]e^x$$
$$= x^n e^x$$

43. Let $w = \sqrt{x}$. Then $dw = \dfrac{dx}{2\sqrt{x}}$, so $dx = 2\sqrt{x}\ dw = 2w\,dw$.

$$\int \sin\sqrt{x}\ dx = \int (\sin w)(2w\ dw) = 2\int w \sin w\ dw$$

Let $u = w \qquad dv = \sin w\ dw$

$du = dw \qquad v = -\cos w$

$$\int w \sin w\ dw = -w\cos w + \int \cos w\ dw = -w\cos w + \sin w + C$$

$$\int \sin\sqrt{x}\ dx = 2\int w \sin w\ dw = -2w\cos w + 2\sin w + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

44. Let $w = \sqrt{3x+9}$. Then $dw = \dfrac{1}{2\sqrt{3x+9}}(3)\,dx$, so

$$dx = \frac{2}{3}\sqrt{3x+9}\ dw = \frac{2}{3}w\,dw.$$

$$\int e^{\sqrt{3x+9}}\ dw = \int (e^w)\left(\frac{2}{3}w\,dw\right) = \frac{2}{3}\int w e^w\ dw$$

Let $u = w \quad dv = e^w\ dw$

$du = dw \quad v = e^w$

$$\int w e^w\ dw = we^w - \int e^w\ dw = we^w - e^w = (w-1)e^w$$

$$\int e^{\sqrt{3x+9}} dx = \frac{2}{3}\int w e^w\ dw = \frac{2}{3}(w-1)e^w = \frac{2}{3}(\sqrt{3x+9} - 1)e^{\sqrt{3x+9}} + C$$

45. Let $w = x^2$. Then $dw = 2x\,dx$.

$$\int x^7 e^{x^2} dx = \int (x^2)^3 e^{x^2} x\,dx = \frac{1}{2}\int w^3\ e^w\ dw.$$

Use tabular integration with $f(x) = w^3$ and $g(w) = e^w$.

$f(w)$ and its derivatives		$g(w)$ and its integrals
w^3	(+)	e^w
$3w^2$	(−)	e^w
$6w$	(+)	e^w
6	(−)	e^w
0		e^w

$$\int w^3 e^w\ dw = w^3 e^w - 3w^2 e^w + 6we^w - 6e^w + C = (w^3 - 3w^2 + 6w - 6)e^w + C$$

$$\int x^7 e^{x^2}\ dx = \frac{1}{2}\int w^3\ e^w\ dw = \frac{1}{2}(w^3 - 3w^2 + 6w - 6)e^w + C = \frac{(x^6 - 3x^4 + 6x^2 - 6)e^{x^2}}{2} + C$$

46. Let $y = \ln r$. Then $dy = \dfrac{1}{r}dr$, and so $dr = r\ dy = e^y\ dy$.

Using the result of Exercise 17, we have:

$$\int \sin(\ln r)\ dr = \int (\sin y)e^y\,dy = \frac{1}{2}e^y(\sin y - \cos y) + C = \frac{1}{2}e^{\ln r}[\sin(\ln r) - \cos(\ln r)] + C = \frac{r}{2}[\sin(\ln r) - \cos(\ln r)] + C$$

47. Let $u = x^n \qquad dv = \cos x\ dx$

$du = nx^{n-1}dx \qquad v = \sin x$

$$\int x^n \cos x\ dx = x^n \sin x - \int (\sin x)(nx^{n-1}dx) = x^n \sin x - n\int x^{n-1}\sin x\ dx$$

48. Let $u = x^n \qquad dv = \sin x\ dx$

$du = nx^{n-1}dx \qquad v = -\cos x$

$$\int n^x \sin x\ dx = (x^n)(-\cos x) - \int (-\cos x)(nx^{n-1})\ dx = -x^n\cos x + n\int x^{n-1}\cos x\ dx$$

49. Let $u = x^n \qquad dv = e^{ax}\ dx$

$du = nx^{n-1}\ dx \qquad v = \dfrac{1}{a}e^{ax}$

$$\int x^n e^{ax}\ dx = (x^n)\left(\frac{1}{a}e^{ax}\right) - \int\left(\frac{1}{a}e^{ax}\right)(nx^{n-1}\ dx) = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\ dx,\ a \neq 0$$

50. Let $u = (\ln x)^n \qquad dv = dx$

$du = \frac{n(\ln x)^{n-1}}{x}\,dx \qquad v = x$

$$\int (\ln x)^n\,dx = (\ln x)^n(x) - \int x\left[\frac{n(\ln x)^{n-1}}{x}\right]dx$$
$$= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

51. (a) Let $y = f^{-1}(x)$. Then $x = f(y)$, so $dx = f'(y)\,dy$.

Hence, $\int f^{-1}(x)\,dx = \int (y)[f'(y)\,dy] = \int y f'(y)\,dy$

(b) Let $u = y \qquad dv = f'(y)dy$

$du = dy \qquad v = f(y)$

$$\int y f'(y)\,dy = y f(y) - \int f(y)\,dy$$
$$= f^{-1}(x)(x) - \int f(y)\,dy$$

Hence, $\int f^{-1}(x)\,dx = \int y f'(y)\,dy$

$$= x f^{-1}(x) - \int f(y)\,dy.$$

52. Let $u = f^{-1}(x) \qquad dv = dx$

$du = \left(\frac{d}{dx}f^{-1}(x)\right)dx \qquad v = x$

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx$$

53. (a) Using $y = f^{-1}(x) = \sin^{-1} x$ and $f(y) = \sin y$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have:

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \sin y\,dy$$
$$= x\sin^{-1}x + \cos y + C$$
$$= x\sin^{-1}x + \cos(\sin^{-1}x) + C$$

(b) $\int \sin^{-1}x\,dx = x\sin^{-1}x - \int x\left(\frac{d}{dx}\sin^{-1}x\right)dx$

$$= x\sin^{-1}x - \int x\frac{1}{\sqrt{1-x^2}}\,dx$$

$u = 1 - x^2,\ du = -2x\,dx$

$$= x\sin^{-1}x + \frac{1}{2}\int u^{-1/2}du$$
$$= x\sin^{-1}x + u^{1/2} + C$$
$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

(c) $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

54. (a) Using $y = f^{-1}(x) = \tan^{-1} x$ and $f(y) = \tan y$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$, we have:

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \tan y\,dy$$
$$= x\tan^{-1}x - \ln|\sec y| + C$$
$$= x\tan^{-1}x + \ln|\cos y| + C$$
$$= x\tan^{-1}x + \ln|\cos(\tan^{-1}x)| + C$$

(b) $\int \tan^{-1}x\,dx = x\tan^{-1}x - \int x\left(\frac{d}{dx}\tan^{-1}x\right)dx$

$$= x\tan^{-1}x - \int x\left(\frac{1}{1+x^2}\right)dx$$

$u = 1 + x^2,\ du = 2x\,dx$

$$= x\tan^{-1}x - \frac{1}{2}\int u^{-1}\,du$$
$$= x\tan^{-1}x - \frac{1}{2}\ln|u| + C$$
$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

(c) $\ln|\cos(\tan^{-1}x)| = \ln\left|\frac{1}{\sqrt{1+x^2}}\right| = -\frac{1}{2}\ln(1+x^2)$

55. (a) Using $y = f^{-1}(x) = \cos^{-1} x$ and $f(y) = \cos x,\ 0 \le x \le \pi$, we have:

$$\int \cos^{-1}x\,dx = x\cos^{-1}x - \int \cos y\,dy$$
$$= x\cos^{-1}x - \sin y + C$$
$$= x\cos^{-1}x - \sin(\cos^{-1}x) + C$$

(b) $\int \cos^{-1}x\,dx = x\cos^{-1}x - \int x\left(\frac{d}{dx}\cos^{-1}x\right)dx$

$$= x\cos^{-1}x - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$u = 1 - x^2,\ du = -2x\,dx$

$$= x\cos^{-1}x - \frac{1}{2}\int u^{-1/2}\,du$$
$$= x\cos^{-1}x - u^{1/2} + C$$
$$= x\cos^{-1}x - \sqrt{1-x^2} + C$$

(c) $\sin(\cos^{-1}x) = \sqrt{1-x^2}$

56. (a) Using $y = f^{-1}(x) = \log_2 x$ and $f(y) = 2^y$, we have

$$\int \log_2 x\,dx = x\log_2 x - \int 2^y\,dy$$
$$= x\log_2 x - \frac{2^y}{\ln 2} + C$$
$$= x\log_2 x - \frac{1}{\ln 2}2^{\log_2 x}$$

(b) $\int \log_2 x\,dx = x\log_2 x - \int x\left(\frac{d}{dx}\log_2 x\right)dx$

$$= x\log_2 x - \int x\left(\frac{1}{x\ln 2}\right)dx$$
$$= x\log_2 x - \int \frac{dx}{\ln 2}$$
$$= x\log_2 x - \left(\frac{1}{\ln 2}\right) + C$$

(c) $2^{\log_2 x} = x$