Chapter 6
Differential Equations and Mathematical Modeling

Section 6.1 Slope Fields and Euler's Method (pp. 321–330)

Exploration 1 Seeing the Slopes

1. Since $\frac{dy}{dx}=0$ represents a line with a slope of 0, we should expect to see horizontal slope lines. We see this at odd multiples of $\pi/2$.
2. The formula $\frac{dy}{dx}=\cos x$ depends only on x, not on y.
3. Yes. The curves are vertical translations of each other, so they all have the same slope at any given value of x.
4. At $x=0$, $\frac{dy}{dx}=\cos 0=1$, so the slope at 0 should be 1. That appears to be the slope of each curve as it crosses the y-axis.
5. At $x=\pi$, $\frac{dy}{dx}=\cos\pi=-1$, so the slope should be -1. That appears to be the slope of each curve at $x=\pi$.
6. Yes. The curves themselves are graphs of odd functions, but we see that the *slopes* at the points (x, y) and $(-x, -y)$ are the same.

Quick Review 6.1

1. Yes. $\frac{d}{dx}e^x=e^x$
2. Yes. $\frac{d}{dx}e^{4x}=4e^{4x}$
3. No. $\frac{d}{dx}(x^2e^x)=2xe^x+x^2e^x$
4. Yes. $\frac{d}{dx}e^{x^2}=2xe^{x^2}$
5. No. $\frac{d}{dx}(e^{x^2}+5)=2xe^{x^2}$
6. Yes. $\frac{d}{dx}\sqrt{2x}=\frac{1}{2\sqrt{2x}}(2)=\frac{1}{\sqrt{2x}}$
7. Yes. $\frac{d}{dx}\sec x=\sec x\tan x$
8. No. $\frac{d}{dx}x^{-1}=-x^{-2}$
9. $y=3x^2+4x+C$
   $2=3(1)^2+4(1)+C$
   $C=-5$
10. $y=2\sin x-3\cos x+C$
    $4=2\sin(0)-3\cos(0)+C$
    $C=-7$
11. $y=e^{2x}+\sec x+C$
    $5=e^{2(0)}+\sec(0)+C$
    $C=3$
12. $y=\tan^{-1}x+\ln(2x-1)+C$
    $\pi=\tan^{-1}(1)+\ln(2(1)-1)+C$
    $C=\frac{3\pi}{4}$

Section 6.1 Exercises

1. $\int dy=\int(5x^4-\sec^2 x)\,dx$
   $y=x^5-\tan x+C$
2. $\int dy=\int(\sec x\tan x-e^x)\,dx$
   $y=\sec x-e^x+C$
3. $\int dy=\int(\sin x-e^{-x}+8x^3)\,dx$
   $y=-\cos x+e^{-x}+2x^4+C$
4. $\int dy=\int\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\ln x+\frac{1}{x}+C$
5. $\int dy=\int\left(5^x\ln 5+\frac{1}{x^2+1}\right)dx=5^x+\tan^{-1}x+C$
6. $\int dy=\int\left(\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{x}}\right)dx=\sin^{-1}x-2\sqrt{x}+C$
7. $\int dy=\int(3t\cos(t^3))\,dt=\sin(t^3)+C$
8. $\int dy=\int \cos t\, e^{\sin t}\,dt$
   $=e^{\sin t}+C$
9. $\int dy=\int(\sec^2(x^5)(5x^4))\,dx$
   $=\tan(x^5)+C$
10. $\int dy=\int 4(\sin u)^3\cos u\,du$
    $=(\sin u)^4+C=\sin^4 u+C$
11. $\int dy=\int 3\sin x\;dx=-3\cos x+C$
    $2=-3\cos(0)+C,\quad C=5$
    $y=-3\cos x+5$
12. $\int dy=\int 2e^x-\cos x\;dx=2e^x-\sin x+C$
    $3=2e^0-\sin(0)+C,\quad C=1$
    $y=2e^x-\sin x+1$
13. $\int du=\int(7x^6-3x^2+5)\,dx=x^7-x^3+5x+C$
    $1=1^7-1^3+5+C,\quad C=-4$
    $u=x^7-x^3+5x-4$

14. $\int dA = \int (10x^9 + 5x^4 - 2x + 4)\,dx = x^{10} + x^5 - x^2 + 4x + C$

$6 = 1^{10} + 1^5 - 1^2 + 4(1) + C, \quad C = 1$

$A = x^{10} + x^5 - x^2 + 4x + 1$

15. $\int dy = \int \left(-\frac{1}{x^2} - \frac{3}{x^4} + 12\right) dx = x^{-1} + x^{-3} + 12x + C$

$3 = 1^{-1} + 1^{-3} + 12(1) + C, \quad C = -11$

$y = x^{-1} + x^{-3} + 12x - 11 \quad (x > 0)$

16. $\int dy = \int \left(5\sec^2 x - \frac{3}{2}\sqrt{x}\right) dx = 5\tan x - x^{3/2} + C$

$7 = 5\tan(0) - (0)^{3/2} + C, \quad C = 7$

$y = 5\tan x - x^{3/2} + 7$

17. $\int dy = \int \left(\frac{1}{1+t^2} + 2^t \ln 2\right) dt = \tan^{-1} t + 2^t + C$

$3 = \tan^{-1}(0) + 2^0 + C, \quad C = 2$

$y = \tan^{-1} t + 2^t + 2$

18. $\int dx = \int \left(\frac{1}{t} - \frac{1}{t^2} + 6\right) dt = \ln t + t^{-1} + 6t + C$

$0 = \ln(1) + 1^{-1} + 6(1) + C, \quad C = -7$

$x = \ln t + t^{-1} + 6t - 7 \qquad (t > 0)$

19. $\int dv = \int \left(4\sec t \tan t + e^t + 6t\right) dt = 4\sec t + e^t + 3t^2 + C$

$5 = 4\sec(0) + e^0 + 3(0)^2 + C, \quad C = 0$

$v = 4\sec t + e^t + 3t^2 \qquad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$

20. $\int ds = \int t(3t - 2)\,dt = t^3 - t^2 + C$

$0 = (1)^3 - (1)^2 + C, \quad C = 0$

$s = t^3 - t^2$

21. $\frac{dy}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\int_1^x \sin(t^2)\,dt$

$y = \int_1^x \sin(t^2)\,dt + 5$

22. $\frac{du}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\int_0^x \sqrt{2 + \cos t}\,dt$

$u = \int_0^x \sqrt{2 + \cos t}\,dt - 3$

23. $F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\int_2^x e^{\cos t}\,dt$

$F(x) = \int_2^x e^{\cos t}\,dt + 9$

24. $G'(s) = \frac{d}{ds}\int_a^s f(t)\,dt = \frac{d}{ds}\int_0^s \sqrt[3]{\tan t}\,dt$

$G(s) = \int_0^s \sqrt[3]{\tan t}\,dt + 4$

25. Graph (b).

$(\sin 0)^2 = 0$

$(\sin 1)^2 > 0$

$(\sin(-1))^2 > 0$

26. Graph (c).

$(\sin 0)^3 = 0$

$(\sin 1)^3 > 0$

$(\sin(-1))^3 < 0$

27. Graph (a).

$(\cos 0)^2 > 0$

$(\cos 1)^2 > 0$

$(\cos(-1)^2 > 0$

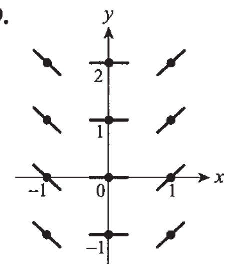
28. Graph (d).

$(\cos 0)^3 > 0$

$(\cos 1)^3 > 0$

$(\cos(-2))^3 < 0$

29.



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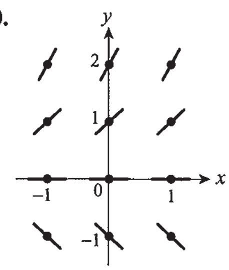

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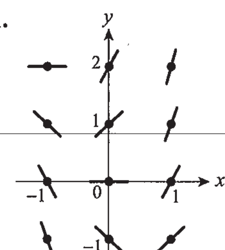

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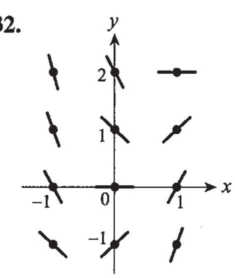

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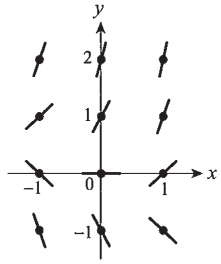
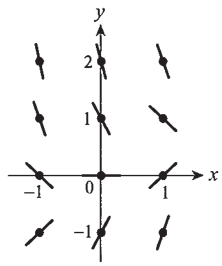
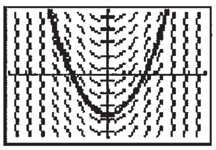


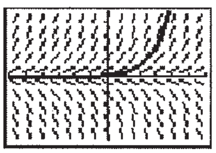
34.



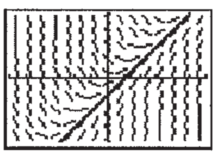
35. (c)



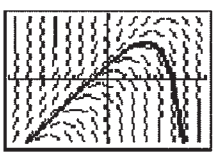
36. (e)



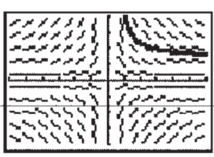
37. (a)



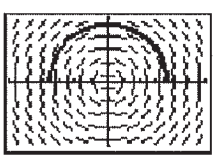
38. (d)



39. (b)



40. (f)



41.

(x, y)	$\frac{dy}{dx} = x - 1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 2)	0.0	0.1	0	(1.1, 2)
(1.1, 2)	0.1	0.1	0.01	(1.2, 2.01)
(1.2, 2.01)	0.2	0.1	0.02	(1.3, 2.03)

$y = 2.03$

42.

(x, y)	$\frac{dy}{dx} = y - 1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 3)	2.0	0.1	0.2	(1.1, 3.2)
(1.1, 3.2)	2.2	0.1	0.22	(1.2, 3.42)
(1.2, 3.42)	2.42	0.1	0.242	(1.3, 3.662)

$y = 3.662$

43.

(x, y)	$\frac{dy}{dx} = y - x$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 2)	1.0	0.1	0.1	(1.1, 2.1)
(1.1, 2.1)	1.0	0.1	0.1	(1.2, 2.2)
(1.2, 2.2)	1.0	0.1	0.1	(1.3, 2.3)

$y = 2.3$

44.

(x, y)	$\frac{dy}{dx} = 2x - y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 0)	2.0	0.1	0.2	(1.1, 0.2)
(1.1, 0.2)	2.0	0.1	0.2	(1.2, 0.4)
(1.2, 0.4)	2.0	0.1	0.2	(1.3, 0.6)

$y = 0.6$

45.

(x, y)	$\frac{dy}{dx} = 2 - x$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 1)	0.0	−0.1	0.0	(1.9, 1)
(1.9, 1)	0.1	−0.1	−0.01	(1.8, 0.99)
(1.8, 0.99)	0.2	−0.1	−0.02	(1.7, 0.97)

$y = 0.97$

46.

(x, y)	$\frac{dy}{dx} = 1 + y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 0)	1.0	−0.1	−0.1	(1.9, −0.1)
(1.9, −0.1)	0.9	−0.1	−0.09	(1.8, −0.19)
(1.8, −0.19)	0.81	−0.1	−0.081	(1.7, −0.271)

$y = -0.271$

47.

(x, y)	$\frac{dy}{dx} = x - y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 2)	−0.0	−0.1	0	(1.9, 2.0)
(1.9, 2)	−0.1	−0.1	0.01	(1.8, 2.01)
(1.8, 2.01)	−0.21	−0.1	0.021	(1.7, 2.031)

$y = 2.031$

48.

(x, y)	$\frac{dy}{dx} = x - 2y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 1)	0.0	−0.1	0.0	(1.9, 1.0)
(1.9, 1)	−0.1	−0.1	0.01	(1.8, 1.01)
(1.8, 1.01)	−0.22	−0.1	0.022	(1.7, 1.032)

$y = 1.032$

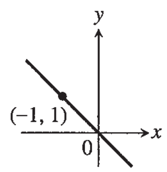
49. **(a)** Graph (b)

(b) The slope is always positive, so (a) and (c) can be ruled out.

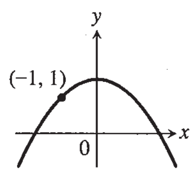
50. **(a)** Graph (b)

(b) The solution should have positive slope when x is negative, zero slope when x is zero and negative slope when x is positive since slope $= \frac{dy}{dx} = -x$.

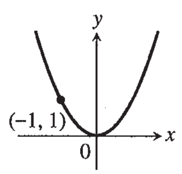
Graphs (a) and (c) don't show this slope pattern.



(a)



(b)



(c)

51. There are positive slopes in the second quadrent of the slope field. The graph of $y = x^2$ has negative slopes in the second quadrent.

52. The slope of $y = \sin x$ would be +1 at the origin, while the slope field shows a slope of zero at every point on the y-axis.

53.

(x, y)	$\frac{dy}{dx} = 2x+1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x+\Delta x, y+\Delta y)$
(1, 3)	3.0	0.1	0.3	(1.1, 3.3)
(1.1, 3.3)	3.2	0.1	0.32	(1.2, 3.62)
(1.2, 3.62)	3.4	0.1	0.34	(1.3, 3.96)
(1.3, 3.96)	3.6	0.1	0.36	(1.4, 4.32)

$y = 4.32$

Euler's Method gives an estimate $f(1.4) \approx 4.32$.
The solution to the initial value problem is $f(x) = x^2 + x + 1$, from which we get $f(1.4) = 4.36$. The percentage error is thus $\frac{4.36-4.32}{4.36} = 0.9\%$.

54.

(x, y)	$\frac{dy}{dx} = 2x - 1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x+\Delta x, y+\Delta y)$
(2, 3)	3.0	−0.1	−0.3	(1.9, 2.7)
(1.9, 2.7)	2.8	−0.1	−0.28	(1.8, 2.42)
(1.8, 2.42)	2.6	−0.1	−0.26	(1.7, 2.16)
1.7, 2.16)	2.4	−0.1	−0.24	(1.6, 1.92)

$y = 1.92$

Euler's Method gives an estimate $f(1.6) \approx 1.92$. The solution to the initial value problem is $f(x) = x^2 - x + 1$, from which we get $f(1.6) = 1.96$. The percentage error is thus $\frac{1.96-1.92}{1.96} = 2\%$.

55. At every (x, y), $(e^{(x-y)/2})(e^{(y-x)/2}) = -e^0 = -1$, so the slopes are negative reciprocals. The slope lines are therefore perpendicular.

56. Since the slopes must be negative reciprocals,
$g(x) = -\frac{1}{\sec x} = -\cos x$.

57. The perpendicular slope field would be produced by $\frac{dy}{dx} = -\sin x$, so $y = \cos x + C$ for any constant C.

58. The perpendicular slope field would be produced by $\frac{dy}{dx} = -x$, so $y = -0.5x^2 + C$ for any constant C.

59. True. They are all lines of the form $y = 5x + C$.

60. False. For example, $f(x) = x^2$ is a solution of $\frac{dy}{dx} = 2x$, but $f^{-1}(x) = \sqrt{x}$ is not a solution of $\frac{dy}{dx} = 2y$.

61. C. For all points with $y = 42$, $m = 42 - 42 = 0$

62. E. $y < 0$, $x^2 > 0$, therefore $\frac{dy}{dx} < 0$.

63. B. $y(0) = e^{0^2} = 1$

$\frac{dy}{dx} = 2xe^{x^2} = 2xy$.

64. A.

65. (a) $\frac{dy}{dx}=x-\frac{1}{x^2}$

$\int \frac{dy}{dx}\,dx=\int (x-x^{-2})\,dx$

$y=\frac{x^2}{2}+x^{-1}+C=\frac{x^2}{2}+\frac{1}{x}+C$

Initial condition: $y(1)=2$

$2=\frac{1^2}{2}+\frac{1}{1}+C$

$2=\frac{3}{2}+C$

$\frac{1}{2}=C$

Solution: $y=\frac{x^2}{2}+\frac{1}{x}+\frac{1}{2}, x>0$

(b) Again, $y=\frac{x^2}{2}+\frac{1}{x}+C.$

Initial condition: $y(-1)=1$

$1=\frac{(-1)^2}{2}+\frac{1}{(-1)}+C$

$1=\frac{-1}{2}+C$

$\frac{3}{2}=C$

Solution: $y=\frac{x^2}{2}+\frac{1}{x}+\frac{3}{2}, x<0$

(c) For $x<0, \frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{x}+\frac{x^2}{2}+C_1\right)$

$=-\frac{1}{x^2}+x$

$=x-\frac{1}{x^2}.$

For $x>0, \frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{x}+\frac{x^2}{2}+C_2\right)$

$=-\frac{1}{x^2}+x$

$=x-\frac{1}{x^2}.$

And for $x=0, \frac{dy}{dx}$ is undefined.

(d) Let C_1 be the value from part (b), and let C_2 be the value from part (a). Thus, $C_1=\frac{3}{2}$ and $C_2=\frac{1}{2}.$

(e) $y(2)=-1$ $\quad$ $y(-2)=2$

$-1=\frac{1}{2}+\frac{2^2}{2}+C_2$ $\quad$ $2=\frac{1}{(-2)}+\frac{(-2)^2}{2}+C_1$

$-1=\frac{5}{2}+C_2$ $\quad$ $2=\frac{3}{2}+C_1$

$-\frac{7}{2}=C_2$ $\quad$ $\frac{1}{2}=C_1$

Thus, $C_1=\frac{1}{2}$ and $C_2=-\frac{7}{2}.$

66. (a) $\frac{d}{dx}(\ln x+C)=\frac{1}{x}$ for $x>0$

(b) $\frac{d}{dx}\left[\ln(-x)+C\right]=\frac{1}{-x}\frac{d}{dx}(-x)=\left(\frac{1}{-x}\right)(-1)=\frac{1}{x}$ for $x<0$

(c) For $x>0$, $\ln|x|+C=\ln x+C$, which is a solution to the differential equation, as we showed in part (a). For $x<0$, $\ln|x|+C=\ln(-x)+C$, which is a solution to the differential equation, as we showed in part (b). Thus, $\frac{d}{dx}\ln|x|=\frac{1}{x}$ for all x except 0.

(d) For $x<0$, we have $y=\ln(-x)+C_1$, which is a solution to the differential equation, as we showed in part (a). For $x>0$, we have $y=\ln x+C_2$, which is a solution to the differential equation, as we showed part (b). Thus, $\frac{dy}{dx}=\frac{1}{x}$ for all x except 0.

67. (a) $y'=\int(12x+4)dx$

$y'=6x^2+4x+C_1$

$y=\int(6x^2+4x+C_1)dx$

$y=2x^3+2x^2+C_1x+C_2$

(b) $y'=\int(e^x+\sin x)dx$

$y'=e^x-\cos x+C_1$

$y=\int(e^x-\cos x+C_1)dx$

$y=e^x-\sin x+C_1x+C_2$

(c) $y'=\int(x^3+x^{-3})dx$

$y'=\frac{x^4}{4}-\frac{1}{2x^2}+C_1$

$y=\int\left(\frac{x^4}{4}-\frac{1}{2x^2}+C_1\right)dx$

$y=\frac{x^5}{20}+\frac{1}{2x}+C_1x+C_2$

68. **(a)** $y' = \int (24x^2 - 10)dx$

$$y' = 8x^3 - 10x + C$$
$$3 = 8(1)^3 - 10(1) + C$$
$$C = 5$$
$$y = \int (8x^3 - 10x + 5)dx$$
$$y = 2x^4 - 5x^2 + 5x + C$$
$$5 = 2(1)^4 - 5(1)^2 + 5(1) + C$$
$$C = 3$$
$$y = 2x^4 - 5x^2 + 5x + 3$$

(b) $y' = \int (\cos x - \sin x)dx$

$$y' = \sin x + \cos x + C$$
$$2 = \sin 0 + \cos 0 + C$$
$$C = 1$$
$$y = \int (\sin x + \cos x + 1)dx$$
$$y = -\cos x + \sin x + x + C$$
$$0 = -\cos 0 + \sin 0 + 0 + C$$
$$C = 1$$
$$y = -\cos x + \sin x + x + 1$$

(c) $y' = \int (e^x - x)dx$

$$y' = e^x - \frac{x^2}{2} + C$$
$$0 = e^0 - \frac{0^2}{2} + C$$
$$C = -1$$
$$y = \int \left(e^x - \frac{x^2}{2} - 1 \right) dx$$
$$y = e^x - \frac{x^3}{6} - x + C$$
$$1 = e^0 - \frac{0^3}{6} - 0 + C$$
$$C = 0$$
$$y = e^x - \frac{x^3}{6} - x$$

69. **(a)** $y' = x$

$$y = \int x\,dx = \frac{x^2}{2} + C$$

(b) $y' = -x$

$$y = \int (-x)dx = -\frac{x^2}{2} + C$$

(c) $y' = y$

$$\frac{d}{dx}(Ce^x) = Ce^x$$
$$y = Ce^x$$

(d) $y' = -y$

$$\frac{d}{dx}(Ce^{-x}) = -Ce^{-x}$$
$$y = Ce^{-x}$$

(e) $y' = xy$

$$\frac{d}{dx}\left(Ce^{x^2/2}\right) = Cxe^{x^2/2}$$
$$y = Ce^{x^2/2}$$

70. **(a)** $y'' = x$

$$y' = \int x\,dx = \frac{x^2}{2} + C_1$$
$$y = \int \left(\frac{x^2}{2} + C_1 \right) dx = \frac{x^3}{6} + C_1 x + C_2$$

(b) $y'' = -x$

$$y' = \int (-x)dx = -\frac{x^2}{2} + C_1$$
$$y = \int \left(-\frac{x^2}{2} + C_1 \right) dx = -\frac{x^3}{6} + C_1 x + C_2$$

(c) $y'' = -\sin x$

$$y' = \int (-\sin x)dx = \cos x + C_1$$
$$y = \int (\cos x + C_1)dx = \sin x + C_1 x + C_2$$

(d) $y'' = y$

$$\frac{d}{dx}\left(C_1 e^x + C_2 e^{-x}\right) = C_1 e^x - C_2 e^{-x} = y'$$
$$\frac{d}{dx}\left(C_1 e^x - C_2 e^{-x}\right) = C_1 e^x + C_2 e^{-x} = y''$$
$$y = C_1 e^x + C_2 e^{-x}$$

(e) $y'' = -y$

$$\frac{d}{dx}(C_1 \sin x + C_2 \cos x) = C_1 \cos x - C_2 \sin x = y'$$
$$\frac{d}{dx}\left(C_1 \cos x - C_2 \sin x\right) = -C_1 \sin x - C_2 \cos x$$
$$y = C_1 \sin x + C_2 \cos x$$

Section 6.2 Antidifferentiation by Substitution (pp. 331–340)

Exploration 1 Are $\int f(u)\,du$ and $\int f(u)\,dx$ the Same Thing?

1. $\int f(u)\,du = \int u^3 du$

$$= \frac{u^4}{4} + C$$

2. $\frac{u^4}{4} = \frac{(x^2)^4}{4} = \frac{x^8}{4}$

3. $f(u) = u^3 = (x^2)^3 = x^6$

$$\int f(u)dx = \int x^6 dx = \frac{x^7}{7}$$

4. No