7. Now the (signed) height of the newly-added vertical segment is $f(x)$.

8. The (signed) area of the segment is $\Delta F = \Delta x \cdot f(x)$, so

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x)$$

Quick Review 5.3

1. $\frac{dy}{dx} = \sin x$

2. $\frac{dy}{dx} = \cos x$

3. $\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x$

4. $\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$

5. $\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$

6. $\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x - 1 = \ln x$

7. $\frac{dy}{dx} = \frac{(n+1)x^n}{n+1} = x^n$

8. $\frac{dy}{dx} = -\frac{1}{(2^x+1)^2} \cdot (\ln 2)2^x = -\frac{2^x \ln 2}{(2^x+1)^2}$

9. $\frac{dy}{dx} = xe^x + e^x$

10. $\frac{dy}{dx} = \frac{1}{x^2+1}$

Section 5.3 Exercises

1. (a) $\int_2^2 g(x)\,dx = 0$

(b) $\int_5^1 g(x)\,dx = -\int_1^5 g(x)\,dx = -8$

(c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx = 3(-4) = -12$

(d) $\int_2^5 f(x)\,dx = \int_2^1 f(x)\,dx + \int_1^5 f(x)\,dx$
$= -\int_1^2 f(x)\,dx + \int_1^5 f(x)\,dx$
$= 4 + 6 = 10$

(e) $\int_1^5 [f(x) - g(x)]\,dx = \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$
$= 6 - 8 = -2$

(f) $\int_1^5 [4f(x) - g(x)]\,dx = \int_1^5 4f(x)\,dx - \int_1^5 g(x)\,dx$
$= 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$
$= 4(6) - 8 = 16$

2. (a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$

(b) $\int_7^9 [f(x) + h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx$
$= 5 + 4 = 9$

(c) $\int_7^9 [2f(x) - 3h(x)]\,dx = \int_7^9 2f(x)\,dx - \int_7^9 3h(x)\,dx$
$= 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx$
$= 2(5) - 3(4) = -2$

(d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = 1$

(e) $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx + \int_9^7 f(x)\,dx$
$= \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx$
$= -1 - 5 = -6$

(f) $\int_9^7 [h(x) - f(x)]\,dx = \int_9^7 h(x)\,dx - \int_9^7 f(x)\,dx$
$= -\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx$
$= -4 + 5 = 1$

3. (a) $\int_1^2 f(u)\,du = 5$

(b) $\int_1^2 \sqrt{3}f(z)\,dz = \sqrt{3}\int_1^2 f(z)\,dz = 5\sqrt{3}$

(c) $\int_2^1 f(t)\,dt = -\int_1^2 f(t)\,dt = -5$

(d) $\int_1^2 [-f(x)]\,dx = -\int_1^2 f(x)\,dx = -5$

4. (a) $\int_0^{-3} g(t)\,dt = -\int_{-3}^0 g(t)\,dt = -\sqrt{2}$

(b) $\int_{-3}^0 g(u)\,du = \sqrt{2}$

(c) $\int_{-3}^0 [-g(x)]\,dx = -\int_{-3}^0 g(x)\,dx = -\sqrt{2}$

(d) $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr = \frac{1}{\sqrt{2}}\int_{-3}^0 g(r)\,dr = 1$

5. (a) $\int_3^4 f(z)\,dz = \int_3^0 f(z)\,dz + \int_0^4 f(z)\,dz$
$= -\int_0^3 f(z)\,dz + \int_0^4 f(z)\,dz$
$= -3 + 7 = 4$

(b) $\int_4^3 f(t)\,dt = \int_4^0 f(t)\,dt + \int_0^3 f(t)\,dt$
$= -\int_0^4 f(t)\,dt + \int_0^3 f(t)\,dt$
$= -7 + 3 = -4$

6. (a) $\int_1^3 h(r)\,dr = \int_1^{-1} h(r)\,dr + \int_{-1}^3 h(r)\,dr$
$= -\int_{-1}^1 h(r)\,dr + \int_{-1}^3 h(r)\,dr = 6$

6. Continued

(b) $-\int_3^1 h(u)\,du = -\int_3^{-1} h(u)\,du - \int_{-1}^1 h(u)\,du$

$= \int_{-1}^3 h(u)\,du - \int_{-1}^1 h(u)\,du = 6$

7. max sin (x^2) = sin (1) on [0, 1]

$\int_0^1 \sin(x^2)\,dx \le \sin(1) < 1$

8. $\max\sqrt{x+8} = 3$ and $\min\sqrt{x+8} = 2\sqrt{2}$ on [0, 1]

$2\sqrt{2} \le \int_0^1 \sqrt{x+8}\,dx \le 3$

9. $(b-a)\min f(x) \ge 0$ on $[a, b]$

$0 \le (b-a)\min f(x) \le \int_a^b f(x)\,dx$

10. $(b-a)\max f(x) \le 0$ on $[a, b]$

$\int_a^b f(x)dx \le (b-a)\max f(x) \le 0$

11. An antiderivative of x^2-1 is $F(x) = \frac{1}{3}x^3 - x$.

$av = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}}(x^2-1)\,dx$

$= \frac{1}{\sqrt{3}}\left[F(\sqrt{3}) - F(0)\right]$

$= \frac{1}{\sqrt{3}}(0-0) = 0$

Find $x = c$ in $[0, \sqrt{3}]$ such that $c^2 - 1 = 0$

$c^2 = 1$

$c = \pm 1$

Since 1 is in $[0, \sqrt{3}]$, $x = 1$.

12. An antiderivative of $-\frac{x^2}{2}$ is $F(x) = -\frac{x^3}{6}$.

$av = \frac{1}{3}\int_0^3\left(-\frac{x^2}{2}\right)dx = \frac{1}{3}[F(3)] - F(0)] = \frac{1}{3}\left(-\frac{9}{2}\right) = -\frac{3}{2}$

Find $x = c$ in [0, 3] such that $-\frac{c^2}{2} = -\frac{3}{2}$.

$c^2 = 3$

$c = \pm\sqrt{3}$

Since $\sqrt{3}$ is in [0, 3], $x = \sqrt{3}$.

13. An antiderivative of $-3x^2 - 1$ is $F(x) = -x^3 - x$.

$av = \frac{1}{1}\int_0^1(-3x^2-1)\,dx = F(1) - F(0) = -2$

Find $x = c$ in [0, 1] such that $-3c^2 - 1 = -2$

$-3c^2 = -1$

$c^2 = \frac{1}{3}$

$c = \pm\frac{1}{\sqrt{3}}$

Since $\frac{1}{\sqrt{3}}$ is in [0, 1], $x = \frac{1}{\sqrt{3}}$.

14. An antiderivative of $(x-1)^2$ is $F(x) = \frac{1}{3}(x-1)^3$.

$av = \frac{1}{3}\int_0^3 (x-1)^2\,dx = \frac{1}{3}[F(3) - F(0)] = \frac{1}{3}\left(\frac{8}{3} + \frac{1}{3}\right) = 1$

Find $x = c$ in [0, 3] such that $(c-1)^2 = 1$.

$c - 1 = \pm 1$

$c = 2$ or $c = 0$.

Since both are in [0, 3], $x = 0$ or $x = 2$.

15. The region between the graph and the x-axis is a triangle of height 3 and base 6, so the area of the region is $\frac{1}{2}(3)(6) = 9$.

$av(f) = \frac{1}{6}\int_{-4}^2 f(x)\,dx = \frac{9}{6} = \frac{3}{2}$.

16. The region between the graph and the x-axis is a rectangle with a half circle of radius 1 cut out. The area of the region is $2(1) - \frac{1}{2}\pi(1)^2 = \frac{4-\pi}{2}$.

$av(f) = \frac{1}{2}\int_{-1}^1 f(t)\,dt = \frac{1}{2}\left(\frac{4-\pi}{2}\right) = \frac{4-\pi}{4}$.

17. There are equal areas above and below the x-axis.

$av(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{2\pi} \bullet 0 = 0$

18. Since tan θ is an odd function, there are equal areas above and below the x-axis.

$av(f) = \frac{1}{\pi/2}\int_{-\pi/4}^{\pi/4} f(\theta)\,d\theta = \frac{2}{\pi} \bullet 0 = 0$

19. $\int_\pi^{2\pi} \sin x\,dx = -\cos(2\pi) + \cos(\pi)$

$= -2$

20. $\int_0^{\pi/2} \cos x\,dx = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1$

21. $\int_0^1 e^x\,dx = e^1 - e^0 = e - 1$

22. $\int_0^{\pi/4} \sec^2 x\,dx = \tan\left(\frac{\pi}{4}\right) - \tan 0 = 1$

23. $\int_1^4 2x\,dx = x^2\Big|_1^4 = 4^2 - 1^2 = 15$

24. $\int_{-1}^2 3x^2\,dx = x^3\Big|_{-1}^2 = 2^3 - (-1)^3 = 9$

25. $\int_{-2}^6 5\,dx = 5x\Big|_{-2}^6 = 5(6) - 5(-2) = 40$

26. $\int_3^7 8\,dx = 8x\Big|_3^7 = 8(7) - 8(3) = 32$

27. $\int_{-1}^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{2}$

28. $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6}$

29. $\int_1^e \frac{1}{x}\,dx = \ln e - \ln 1 = 1$

30. $\int_1^4 -x^{-2}\,dx = \frac{1}{x}\Big|_1^4 = \frac{1}{4} - \frac{1}{1} = -\frac{3}{4}$

31. $av(f) = \frac{1}{\pi - 0}\int_0^\pi \sin x\,dx$
$= \frac{1}{\pi}(-\cos\pi - (-\cos 0)) = \frac{2}{\pi}$

32. $av(f) = \frac{1}{2e-e}\int_e^{2e}\frac{1}{x}\,dx = \frac{1}{e}(\ln 2e - \ln e)$
$= \frac{\ln 2}{e}$

33. $av(f) = \frac{1}{\frac{\pi}{4} - 0}\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \tan\left(\frac{\pi}{4}\right) - \tan(0)$
$= \frac{4}{\pi}$

34. $av(f) = \frac{1}{1-0}\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(1) - \tan^{-1}(0)$
$= \frac{\pi}{4}$

35. $av(f) = \frac{1}{2-(-1)}\int_{-1}^2 \left(3x^2 + 2x\right)dx = \frac{1}{3}\left(x^3 + x^2\right)\Big|_{-1}^2$
$= 4$

36. $av(f) = \frac{1}{\frac{\pi}{3} - 0}\int_0^{\frac{\pi}{3}} \sec x \tan x\,dx = \frac{3}{\pi}\left(\sec\left(\frac{\pi}{3}\right) - \sec 0\right)$
$= \frac{3}{\pi}$

37. $\min f = \frac{1}{2}$ and $\max f = 1$

$\frac{1}{2} \le \int_0^1 \frac{1}{1+x^4}\,dx \le 1$

38. $f(0.5) = \frac{16}{17}$

$\left(\frac{1}{2}\right)\left(\frac{16}{17}\right) \le \int_0^{0.5}\frac{1}{1+x^4}\,dx \le \left(\frac{1}{2}\right)(1)$

$\frac{8}{17} \le \int_0^{0.5}\frac{1}{1+x^4}\,dx \le \frac{1}{2}$

$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \le \int_{0.5}^1 \frac{1}{1+x^4}\,dx \le \left(\frac{1}{2}\right)\left(\frac{16}{17}\right)$

$\frac{1}{4} \le \int_{0.5}^1 \frac{1}{1+x^4}\,dx \le \frac{8}{17}$

$\frac{8}{17} + \frac{1}{4} \le \int_0^1 \frac{1}{1+x^4}\,dx \le \frac{1}{2} + \frac{8}{17}$

$\frac{49}{68} \le \int_0^1 \frac{1}{1+x^4}\,dx \le \frac{33}{34}$

39. Yes, $\int_a^b av(f)\,dx = \int_a^b f(x)\,dx$.

This is because $av(f)$ is a constant, so

$$\begin{aligned}\int_a^b av(f)\,dx &= [av(f)\cdot x]_a^b \\ &= av(f)\cdot b - av(f)\cdot a \\ &= (b-a)av(f) \\ &= (b-a)\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right] \\ &= \int_a^b f(x)\,dx\end{aligned}$$

40. **(a)** 300 mi

(b) $\frac{150\text{ mi}}{30\text{ mph}} + \frac{150\text{ mi}}{50\text{ mph}} = 8\text{ h}$

(c) $\frac{300\text{ mi}}{8\text{ h}} = 37.5\text{ mph}$

(d) The average speed is the total distance divided by the total time. Algebraically, $\frac{d_1 + d_2}{t_1 + t_2}$. The driver computed $\frac{1}{2}\left(\frac{d_1}{t_1} + \frac{d_2}{t_2}\right)$. The two expressions are not equal.

41. Time for first release $= \frac{1000\text{ m}^3}{10\text{ m}^3\text{/min}} = 100$ min

Time for second release $= \frac{100\text{ m}^3}{20\text{ m}^3\text{/min}} = 50$ min

Average rate $= \frac{\text{total released}}{\text{total time}} = \frac{2000\text{ m}^3}{150\text{ min}} = 13\frac{1}{3}\text{m}^3\text{/min}$

42. $\int_0^1 \sin x\,dx \le \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$

43. $\int_0^1 \sec x\,dx \ge \int_0^1\left(1 + \frac{x^2}{2}\right)dx = \left[x + \frac{x^3}{6}\right]_0^1 = \frac{7}{6}$

44. Let $L(x) = cx + d$. Then the average value of f on $[a, b]$ is

$$\begin{aligned}av(f) &= \frac{1}{b-a}\int_a^b (cx+d)dx \\ &= \frac{1}{b-a}\left[\left(\frac{cb^2}{2} + db\right) - \left(\frac{ca^2}{2} + da\right)\right] \\ &= \frac{1}{b-a}\left[\frac{c(b^2-a^2)}{2} + d(b-a)\right] \\ &= \frac{c(b+a)+2d}{2} \\ &= \frac{(ca+d)+(cb+d)}{2} \\ &= \frac{L(a)+L(b)}{2}\end{aligned}$$

45. False. For example, $\sin 0 = \sin\pi = 0$, but the average value of $\sin x$ on $[0, \pi]$ is greater than 0.

46. False. For example, $\int_{-3}^3 2x\,dx = 0$ but $2(-3) \ne 2(3)$

47. A. There is no rule for the multiplication of functions.

48. D. There is no rule for the negation of the bounds.

49. B. $av(f) = \dfrac{1}{5-1}\int_1^5 \cos x\,dx = \dfrac{1}{4}(\sin 5 - \sin 1)$
$= -0.450.$

50. C. $10 = \dfrac{1}{b-a}\int_a^b F(x)dx$

$10(b-a) = \int_a^b f(x)\,dx$

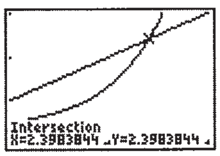
51. **(a)** Area $= \dfrac{1}{2}$bh

(b) $\dfrac{h}{2b}x^2 + C$

(c) $\int_0^b y(x)\,dx = \left[\dfrac{h}{2b}x^2\right]_0^b = \dfrac{hb^2}{2b} = \dfrac{1}{2}bh$

52. $av(x^k) = \dfrac{1}{k}\int_0^k x^k dx = \dfrac{1}{k}\left[\dfrac{1}{k+1}x^{k+1}\right]_0^k = \dfrac{k^{k+1}}{k(k+1)}$

Graph $y_1 = \dfrac{x^{x+1}}{x(x+1)}$ and $y_2 = x$ on a graphing calculator and find the point of intersection for $x > 1$.



[1, 3] by [0, 3]

Thus, $k \approx 2.39838$

53. An antiderivative of $F'(x)$ is $F(x)$ and an antiderivative of $G'(x)$ is $G(x)$.

$\int_a^b F(x)\,dx = F(b) - F(a)$

$\int_a^b G'(x)\,dx = G(b) - G(a)$

Since $F'(x) = G'(x)$, $\int_a^b F'(x)\,dx = \int_a^b G'(x)\,dx$, so $F(b) - F(a) = G(b) - G(a)$.

Quick Quiz Sections 5.1–5.3

1. D. $\int_a^b (F(x)+3)\,dx = a + 2b + \int_a^b 3dx$

$a + 2b + 3b - 3a = 5b - 2a$

2. B.

3. C. $\int_2^2 x^2 dx = \left.\dfrac{x^3}{3}\right|_2^2 = \dfrac{2^3}{3} - \dfrac{2^3}{3} = 0.$

4. **(a)** $f''(x) = 6x + 12$

$\int f''(x)dx = 3x^2 + 12x + c$

$y = 4x - 5$

$m = f' = 4$

$3(0)^2 + 12(0) + c = 4$

$c = 4$

$\int f'(x)dx = \int (3x^2 + 12x + 4)dx$

$f(x) = x^3 + 6x^2 + 4x + c$

$f(0) = (0)^3 + 6(0)^2 + 4(0) + c = -5$

$c = -5$

$f(x) = x^3 + 6x^2 + 4x - 5$

(b) $av(f) = \dfrac{1}{1-(-1)}\int_{-1}^1 (x^3 + 6x^2 + 4x - 5)dx$

$\dfrac{1}{2}\left(\dfrac{x^4}{4} + 2x^3 + 2x^2 - 5x\right)\Bigg|_{-1}^{1} = -3$

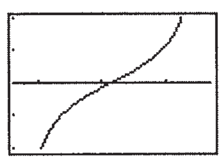
Section 5.4 Fundamental Theorem of Calculus (pp. 294–305)

Exploration 1 Graphing NINT *f*

2. The function $y = \tan x$ has vertical asymptotes at all odd multiples of $\dfrac{\pi}{2}$. There are six of these between -10 and 10.

3. In attempting to find $F(-10) = \int_3^{-10} \tan(t)\,dt + 5$, the calculator must find a limit of Riemann sums for the integral, using values of $\tan t$ for t between -10 and 3. The large positive and negative values of $\tan t$ found near the asymptotes cause the sums to fluctuate erratically so that no limit is approached. (We will see in Section 8.3 that the "areas" near the asymptotes are infinite, although NINT is not designed to determine this.)

4. $y = \tan x$



[1.6, 4.7] by [−2, 2]

5. The domain of this continuous function is the open interval $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

6. The domain of F is the same as the domain of the continuous function in step 4, namely $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.