10. Continued

Since $y = x + 3$, the solutions are:

$x = 0$ and $y = 3$, or, $x = -\frac{24}{13}$ and $y\frac{15}{13}$.

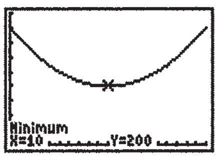
In ordered pair notation, the solution are (0, 3) and $\left(-\frac{24}{13}, \frac{15}{13}\right)$.

Section 4.4 Exercises

1. Represent the numbers by x and $20 - x$, where $0 \le x \le 20$.

(a) The sum of the squares is given by $f(x) = x^2 + (20 - x)^2 = 2x^2 - 40x + 400$. Then $f'(x) = 4x - 40$. The critical point and endpoints occur at $x = 0, x = 10,$ and $x = 20$. Then $f(0) = 400, f(10) = 200$, and $f(20) = 400$. The sum of the squares is as large as possible for the numbers 0 and 20, and is as small as possible for the numbers 10 and 10.

Graphical support:



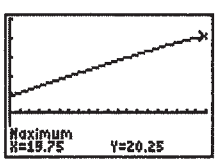
[0, 20] by [0, 450]

(b) The sum of one number plus the square root of the other is given by $g(x) = x + \sqrt{20 - x}$. Then

$g'(x) = 1 - \frac{1}{2\sqrt{20 - x}}$. The critical point occurs when

$2\sqrt{20 - x} = 1$, so $20 - x = \frac{1}{4}$ and $x = \frac{79}{4}$. Testing the endpoints and critical point, we find $g(0) = \sqrt{20} \approx 4.47$, $g\left(\frac{79}{4}\right) = \frac{81}{4} = 20.25$, and $g(20) = 20$. The sum $x + \sqrt{20 - x}$ is as large as possible when the numbers are $x = \frac{79}{4}$ and $20 - x = \frac{1}{4}$. The sum $x + \sqrt{20 - x}$ is as small as possible when the numbers are $x = 0$ and $20 - x = 20$.

Graphical support:



[0, 20] by [−10, 25]

2. Let x and y represent the legs of the triangle, and note that $0 < x < 5$. Then $x^2 + y^2 = 25$, so $y = \sqrt{25 - x^2}$ (since $y > 0$). The area is $A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{25 - x^2}$,

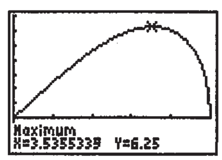
so $\frac{dA}{dx} = \frac{1}{2}x\frac{1}{2\sqrt{25 - x^2}}(-2x) + \frac{1}{2}\sqrt{25 - x^2}$
$= \frac{25 - 2x^2}{2\sqrt{25 - x^2}}$.

The critical point occurs when $25 - 2x^2 = 0$, which means $x = \frac{5}{\sqrt{2}}$, (since $x > 0$). This value corresponds to the largest possible area, since $\frac{dA}{dx} > 0$ for $0 < x < \frac{5}{\sqrt{2}}$ and $\frac{dA}{dx} < 0$ for $\frac{5}{\sqrt{2}} < x < 5$. When $x = \frac{5}{\sqrt{2}}$, we have

$y = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2} = \frac{5}{\sqrt{2}}$ and $A = \frac{1}{2}xy = \frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)^2 = \frac{25}{4}$.

Thus, the largest possible area is $\frac{25}{4}\text{cm}^2$, and the dimensions (legs) are $\frac{5}{\sqrt{2}}$ cm by $\frac{5}{\sqrt{2}}$ cm.

Graphical support:



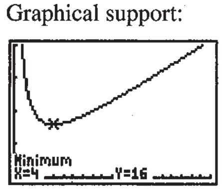
[0, 5] by [−2, 7]

3. Let x represent the length of the rectangle in inches ($x > 0$). Then the width is $\frac{16}{x}$ and the perimeter is

$P(x) = 2\left(x + \frac{16}{x}\right) = 2x + \frac{32}{x}$.

Since $P'(x) = 2 - 32x^{-2} = \frac{2(x^2 - 16)}{x^2}$ this critical point occurs at $x = 4$. Since $P'(x) < 0$ for $0 < x < 4$ and $P'(x) > 0$ for $x > 4$, this critical point corresponds to the minimum perimeter. The smallest possible perimeter is $P(4) = 16$ in., and the rectangle's dimensions are 4 in. by 4 in.

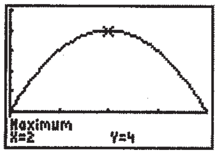
Graphical support:



[0, 20] by [0, 40]

4. Let x represent the length of the rectangle in meters $(0 < x < 4)$. Then the width is $4 - x$ and the area is $A(x) = x(4 - x) = 4x - x^2$. Since $A'(x) = 4 - 2x$, the critical point occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < 4$, this critical point corresponds to the maximum area. The rectangle with the largest area measures 2 m by $4 - 2 = 2$m, so it is a square.

Graphical support:



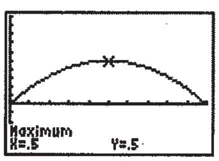
[0, 4] by [−1.5, 5]

5. (a) The equation of line AB is $y = -x + 1$, so the y-coordinate of P is $-x + 1$.

(b) $A(x) = 2x(1 - x)$

(c) Since $A'(x) = \frac{d}{dx}(2x - 2x^2) = 2 - 4x$, the critical point occurs at $x = \frac{1}{2}$. Since $A'(x) > 0$ for $0 < x < \frac{1}{2}$ and $A'(x) < 0$ for $\frac{1}{2} < x < 1$, this critical point corresponds to the maximum area. The largest possible area is $A\left(\frac{1}{2}\right) = \frac{1}{2}$ square unit, and the dimensions of the rectangle are $\frac{1}{2}$ unit by 1 unit.

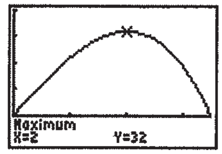
Graphical support:



[0, 1] by [−0.5, 1]

6. If the upper right vertex of the rectangle is located at $(x, 12 - x^2)$ for $0 < x < \sqrt{12}$, then the rectangle's dimensions are $2x$ by $12 - x^2$ and the area is $A(x) = 2x(12 - x^2) = 24x - 2x^3$. Then $A'(x) = 24 - 6x^2 = 6(4 - x^2)$, so the critical point (for $0 < x < \sqrt{12}$) occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < \sqrt{12}$, this critical point corresponds to the maximum area. The largest possible area is $A(2) = 32$, and the dimensions are 4 by 8.

Graphical support:



[0, $\sqrt{12}$] by [−10, 40]

7. Let x be the side length of the cut-out square $(0 < x < 4)$. Then the base measures $8 - 2x$ in. by $15 - 2x$ in., and the volume is

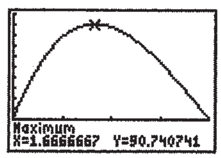
$V(x) = x(8 - 2x)(15 - 2x) = 4x^3 - 46x^2 + 120x$. Then

$V'(x) = 12x^2 - 92x + 120 = 4(3x - 5)(x - 6)$.

Then the critical point (in $0 < x < 4$) occurs at $x = \frac{5}{3}$. Since $V'(x) > 0$ for $0 < x < \frac{5}{3}$ and $V'(x) < 0$ for $\frac{5}{3} < x < 4$, the critical point corresponds to the maximum volume.

The maximum volume is $V\left(\frac{5}{3}\right) = \frac{2450}{27} \approx 90.74\,\text{in}^3$, and the dimensions are $\frac{5}{3}$ in. by $\frac{14}{3}$ in. by $\frac{35}{3}$ in.

Graphical support:



[0, 4] by [−25, 100]

8. Note that the values a and b must satisfy $a^2 + b^2 = 20^2$ and so $b = \sqrt{400 - a^2}$. Then the area is given by

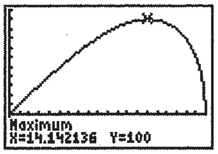
$A = \frac{1}{2}ab = \frac{1}{2}a\sqrt{400 - a^2}$ for $0 < a < 20$, and

$$\frac{dA}{da} = \frac{1}{2}a\left(\frac{1}{2\sqrt{400 - a^2}}\right)(-2a) + \frac{1}{2}\sqrt{400 - a^2}$$

$$= \frac{-a^2 + (400 - a^2)}{2\sqrt{400 - a^2}} = \frac{200 - a^2}{\sqrt{400 - a^2}}.$$ The critical point occurs when $a^2 = 200$. Since $\frac{dA}{da} > 0$ for $0 < a < \sqrt{200}$ and $\frac{dA}{da} < 0$ for $\sqrt{200} < a < 20$, this critical point corresponds to the maximum area. Furthermore, if $a = \sqrt{200}$ then $b = \sqrt{400 - a^2} = \sqrt{200}$, so the maximum area occurs when $a = b$.

8. Continued

Graphical support:



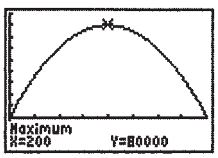
[0, 20] by [−30, 110]

9. Let x be the length in meters of each side that adjoins the river. Then the side parallel to the river measures $800 - 2x$ meters and the area is

$A(x) = x(800 - 2x) = 800x - 2x^2$ for $0 < x < 400$.

Therefore, $A'(x) = 800 - 4x$ and the critical point occurs at $x = 200$. Since $A'(x) > 0$ for $0 < x < 200$ and $A'(x) < 0$ for $200 < x < 400$, the critical point corresponds to the maximum area. The largest possible area is $A(200) = 80{,}000 \text{ m}^2$ and the dimensions are 200 m (perpendicular to the river) by 400 m (parallel to the river).

Graphical support:

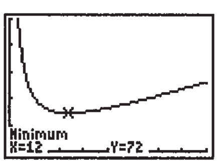


[0, 400] by [−25,000, 90,000]

10. If the subdividing fence measures x meters, then the pea patch measures x m by $\frac{216}{x}$ m and the amount of fence needed is $f(x) = 3x + 2\left(\frac{216}{x}\right) = 3x + 432x^{-1}$. Then

$f'(x) = 3 - 432x^{-2}$ and the critical point (for $x > 0$) occurs at $x = 12$. Since $f'(x) < 0$ for $0 < x < 12$ and $f'(x) > 0$ for $x > 12$, the critical point corresponds to the minimum total length of fence. The pea patch will measure 12 m by 18 m (with a 12-m divider), and the total amount of fence needed is $f(12) = 72\text{ m}$.

Graphical support:



[0, 40] by [0, 250]

11. (a) Let x be the length in feet of each side of the square base. Then the height is $\frac{500}{x^2}$ ft and the surface area (not including the open top) is

$S(x) = x^2 + 4x\left(\frac{500}{x^2}\right) = x^2 + 2000x^{-1}$. Therefore,

$S'(x) = 2x - 2000x^{-2} = \frac{2(x^3 - 1000)}{x^2}$ and the critical point occurs at $x = 10$. Since $S'(x) < 0$ for $0 < x < 10$ and $S'(x) > 0$ for $x > 10$, the critical point corresponds to the minimum amount of steel used. The dimensions should be 10 ft by 10 ft by 5 ft, where the height is 5 ft.

(b) Assume that the weight is minimized when the total area of the bottom and the four sides is minimized.

12. (a) Note that $x^2y = 1125$, so $y = \frac{1125}{x^2}$. Then

$$\begin{aligned} c &= 5(x^2 + 4xy) + 10xy \\ &= 5x^2 + 30xy \\ &= 5x^2 + 30x\left(\frac{1125}{x^2}\right) \\ &= 5x^2 + 33{,}750x^{-1} \end{aligned}$$

$$\frac{dc}{dx} = 10x - 33{,}750x^{-2} = \frac{10(x^3 - 3375)}{x^2}$$

The critical point occurs at $x = 15$. Since $\frac{dc}{dx} < 0$ for $0 < x < 15$ and $\frac{dc}{dx} > 0$ for $x > 15$, the critical point corresponds to the minimum cost. The values of x and y are $x = 15$ ft and $y = 5$ ft.

(b) The material for the tank costs 5 dollars/sq ft and the excavation charge is 10 dollars for each square foot of the cross-sectional area of one wall of the hole.

13. Let x be the height in inches of the printed area. Then the width of the printed area is $\frac{50}{x}$ in. and the overall dimensions are $x + 8$ in. by $\frac{50}{x} + 4$ in. The amount of paper used is $A(x) = (x + 8)\left(\frac{50}{x} + 4\right) = 4x + 82 + \frac{400}{x}$ in^2. Then

$A'(x) = 4 - 400x^{-2} = \frac{4(x^2 - 100)}{x^2}$ and the critical point (for $x > 0$) occurs at $x = 10$. Since $A'(x) < 0$ for $0 < x < 10$ and $A'(x) > 0$ for $x > 10$, the critical point corresponds to the minimum amount of paper. Using $x + 8$ and $\frac{50}{x} + 4$ for $x = 10$, the overall dimensions are 18 in. high by 9 in. wide.

14. **(a)** $s(t) = -16t^2 + 96t + 112$

$v(t) = s'(t) = -32t + 96$

At $t = 0$, the velocity is $v(0) = 96$ ft/sec.

(b) The maximum height occurs when $v(t) = 0$, when $t = 3$. The maximum height is $s(3) = 256$ ft and it occurs at $t = 3$ sec.

(c) Note that $s(t) = -16t^2 + 96t + 112 = -16(t+1)(t-7)$, so $s = 0$ at $t = -1$ or $t = 7$. Choosing the positive value, of t, the velocity when $s = 0$ is $v(7) = -128$ ft/sec.

15. We assume that a and b are held constant. Then $A(\theta) = \frac{1}{2}ab \sin\theta$ and $A'(\theta) = \frac{1}{2}ab\cos\theta$. The critical point (for $0 < \theta < \pi$) occurs at $\theta = \frac{\pi}{2}$. Since $A'(\theta) > 0$ for $0 < \theta < \frac{\pi}{2}$ and $A'(\theta) < 0$ for $\frac{\pi}{2} < \theta < \pi$, the critical point corresponds to the maximum area. The angle that maximizes the triangle's area is $\theta = \frac{\pi}{2}$ (or 90°).

16. Let the can have radius r cm and height h cm. Then $\pi r^2 h = 1000$, so $h = \frac{1000}{\pi r^2}$. The area of material used is

$A = \pi r^2 + 2\pi rh = \pi r^2 + \frac{2000}{r}$, so $\frac{dA}{dr} = 2\pi r - 2000r^{-2}$

$= \frac{2\pi r^3 - 2000}{r^2}$. The critical point occurs at

$r = \sqrt[3]{\frac{1000}{\pi}} = 10\pi^{-1/3}$ cm. Since $\frac{dA}{dr} < 0$ for $0 < r < 10\pi^{-1/3}$ and $\frac{dA}{dr} > 0$ for $r > 10\pi^{1/3}$, the critical point corresponds to the least amount of material used and hence the lightest possible can. The dimensions are $r = 10\pi^{-1/3} \approx 6.83$ cm and $h = 10\pi^{-1/3} \approx 6.83$ cm. In Example 4, because of the top of the can, the "best" design is less big around and taller.

17. Note that $\pi r^2 h = 1000$, so $h = \frac{1000}{\pi r^2}$. Then

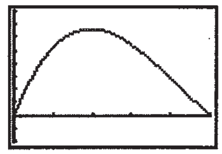
$A = 8r^2 + 2\pi rh = 8r^2 + \frac{2000}{r}$, so

$\frac{dA}{dr} = 16r - 2000r^{-2} = \frac{16(r^3 - 125)}{r^2}$. The critical point occurs at $r = \sqrt[3]{125} = 5$ cm. Since $\frac{dA}{dr} < 0$ for $0 < r < 5$ and $\frac{dA}{dr} > 0$ for $r > 5$, the critical point corresponds to the least amount of aluminium used or wasted and hence the most economical can. The dimensions are $r = 5$ cm and $h = \frac{40}{\pi}$, so the ratio of h to r is $\frac{8}{\pi}$ to 1.

18. **(a)** The base measures $10 - 2x$ in. by $\frac{15 - 2x}{2}$ in, so the volume formula is

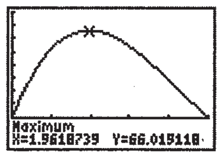
$V(x) = \frac{x(10-2x)(15-2x)}{2} = 2x^3 - 25x^2 + 75x$.

(b) We require $x > 0$, $2x < 10$, and $2x < 15$. Combining these requirements, the domain is the interval (0, 5).



[0, 5] by [−20, 80]

(c)



[0, 5] by [−20, 80]

The maximum volume is approximately 66.02 when $x \approx 1.96$ in.

(d) $V'(x) = 6x^2 - 50x + 75$

The critical point occurs when $V'(x) = 0$, at

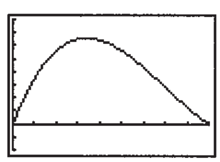
$$x = \frac{50 \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)} = \frac{50 \pm \sqrt{700}}{12} = \frac{25 \pm 5\sqrt{7}}{6},$$

that is, $x \approx 1.96$ or $x \approx 6.37$. We discard the larger value because it is not in the domain. Since $V''(x) = 12x - 50$, which is negative when $x \approx 1.96$, the critical point corresponds to the maximum volume. The maximum volume occurs when $x = \frac{25 - 5\sqrt{7}}{6} \approx 1.96$, which confirms the result in (c).

19. **(a)** The "sides" of the suitcase will measure $24 - 2x$ in. by $18 - 2x$ in. and will be $2x$ in. apart, so the volume formula is

$V(x) = 2x(24 - 2x)(18 - 2x) = 8x^3 - 168x^2 + 864x$.

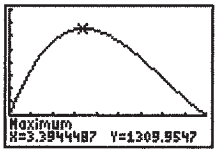
(b) We require $x > 0$, $2x < 18$, and $2x < 24$. Combining these requirements, the domain is the interval (0, 9).



[0, 9] by [−400, 1600]

19. Continued

(c)



[0, 9] by [−400, 1600]

The maximum volume is approximately 1309.95 when $x \approx 3.39$ in.

(d) $V'(x) = 24x^2 - 336x + 864 = 24(x^2 - 14x + 36)$

The critical point is at

$$x = \frac{14 \pm \sqrt{(-14)^2 - 4(1)(36)}}{2(1)} = \frac{14 \pm \sqrt{52}}{2} = 7 \pm \sqrt{13}, \text{ that}$$

is, $x \approx 3.39$ or $x \approx 10.61$. We discard the larger value because it is not in the domain. Since $V''(x) = 24(2x - 14)$, which is negative when $x \approx 3.39$, the critical point corresponds to the maximum volume. The maximum value occurs at $x = 7 - \sqrt{13} \approx 3.39$, which confirms the results in (c).

(e) $8x^3 - 168x^2 + 864x = 1120$

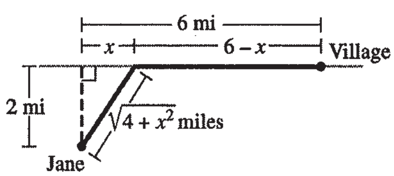
$8(x^3 - 21x^2 + 108x - 140) = 0$

$8(x-2)(x-5)(x-14) = 0$

Since 14 is not in the domain, the possible values of x are $x = 2$ in. or $x = 5$ in.

(f) The dimensions of the resulting box are $2x$ in., $(24 - 2x)$ in., and $(18 - 2x)$ in. Each of these measurements must be positive, so that gives the domain of (0, 9)

20.



Let x be the distance from the point on the shoreline nearest Jane's boat to the point where she lands her boat. Then she needs to row $\sqrt{4+x^2}$ mi at 2 mph and walk $6 - x$ mi at 5 mph. The total amount of time to reach the village is

$$f(x) = \frac{\sqrt{4+x^2}}{2} + \frac{6-x}{5} \text{ hours } (0 \le x \le 6). \text{ Then}$$

$$f'(x) = \frac{1}{2}\frac{1}{2\sqrt{4+x^2}}(2x) - \frac{1}{5} = \frac{x}{2\sqrt{4+x^2}} - \frac{1}{5}.$$

Solving $f'(x) = 0$, we have:

$$\frac{x}{2\sqrt{4+x^2}} = \frac{1}{5}$$

$$5x = 2\sqrt{4+x^2}$$

$$25x^2 = 4(4+x^2)$$

$$21x^2 = 16$$

$$x = \pm\frac{4}{\sqrt{21}}$$

We discard the negative value of x because it is not in the domain. Checking the endpoints and critical point, we have $f(0) = 2.2, f\left(\frac{4}{\sqrt{21}}\right) \approx 2.12$, and $f(6) \approx 3.16$. Jane should land her boat $\frac{4}{\sqrt{21}} \approx 0.87$ miles down the shoreline from the point nearest her boat.

21. If the upper right vertex of the rectangle is located at $(x, 4 \cos 0.5x)$ for $0 < x < \pi$, then the rectangle has width $2x$ and height $4 \cos 0.5x$, so the area is $A(x) = 8x \cos 0.5x$. Then $A'(x) = 8\,x(-0.5 \sin 0.5x) + 8(\cos 0.5x)(1)$
$= -4x \sin 0.5x + 8 \cos 0.5x$.
Solving $A'(x)$ graphically for $0 < x < \pi$, we find that $x \approx 1.72$. Evaluating $2x$ and $4 \cos 0.5x$ for $x \approx 1.72$, the dimensions of the rectangle are approximately 3.44 (width) by 2.61 (height), and the maximum area is approximately 8.98.

22. Let the radius of the cylinder be r cm, $0 < r < 10$. Then the height is $2\sqrt{100 - r^2}$ and the volume is $V(r) = 2\pi r^2\sqrt{100 - r^2}\,\text{cm}^3$. Then

$$V'(r) = 2\pi r^2\left(\frac{1}{2\sqrt{100-r^2}}\right)(-2r) + (2\pi\sqrt{100-r^2})(2r)$$

$$= \frac{-2\pi r^3 + 4\pi r(100 - r^2)}{\sqrt{100-r^2}}$$

$$= \frac{2\pi r(200 - 3r^2)}{\sqrt{100-r^2}}$$

The critical point for $0 < r < 10$ occurs at $r = \sqrt{\frac{200}{3}} = 10\sqrt{\frac{2}{3}}$. Since $V'(r) > 0$ for $0 < r < 10\sqrt{\frac{2}{3}}$ and $V'(r) > 0$ for $10\sqrt{\frac{2}{3}} < r < 10$, the critical point corresponds to the maximum volume. The dimensions are $r = 10\sqrt{\frac{2}{3}} \approx 8.16$ cm and $h = \frac{20}{\sqrt{3}} \approx 11.55$ cm, and the volume is $\frac{4000\pi}{3\sqrt{3}} \approx 2418.40\ \text{cm}^3$.

23. Set $r'(x) = c'(x)$: $4x^{-1/2} = 4x$. The only positive critical value is $x = 1$, so profit is maximized at a production level of 1000 units. Note that $(r-c)''(x) = -2(x)^{-3/2} - 4 < 0$ for all positive x, so the Second Derivative Test confirms the Maximum.

24. Set $r'(x) = c'(x)$: $2x/(x^2+1)^2 = (x-1)^2$. We solve this equation grpahically to find that $x \approx 0.294$. The graph of $y = r(x) - c(x)$ shows a minimum at $x \approx 0.294$ and a maximum at $x \approx 1.525$, so profit is maximized at a production level of about 1,525 units.

25. Set $\bar{c}(x) = \dfrac{c(x)}{x} = x^2 - 10x + 30$. The only positive solution to $\dfrac{d}{dx}\left(\dfrac{c(x)}{x}\right) = 0$ is $x = 5$, so average cost is minimized at a production level of 5000 units. Note that $\dfrac{d^2}{dx^2}\left(\dfrac{c(x)}{x}\right) = 2 > 0$ for all positive x, so the Second Derivative Test Confirms the minimum.

26. Set $\bar{c}(x) = \dfrac{c(x)}{x} = e^x - 2x$. The only positive solution to $\dfrac{d}{dx}\left(\dfrac{c(x)}{x}\right) = 0$ is $x = \ln 2$, so average cost is minimized at a production level of 1000 ln 2, which is about 693 units. Note that $\dfrac{d^2}{dx^2}\left(\dfrac{c(x)}{x}\right) = e^x > 0$ for all positive x, so the Second Derivative Test confirms the minimum.

27. Revenue: $r(x) = [200 - 2(x-50)]x = -2x^2 + 300x$

Cost: $c(x) = 6000 + 32x$

Profit: $p(x) = r(x) - c(x)$

$= -2x^2 + 268x - 6000, 50 \le x \le 80$

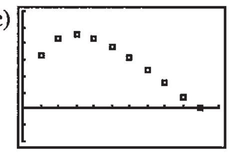
Since $p'(x) = -4x + 268 = -4(x-67)$, the critical point occurs at $x = 67$. This value represents the maximum because $p''(x) = -4$, which is negative for all x in the domain. The maximum profit occurs if 67 people go on the tour.

28. **(a)** $f'(x) = x(-e^{-x}) + e^{-x}(1) = e^{-x}(1-x)$

The critical point occurs at $x = 1$. Since $f'(x) > 0$ for $0 \le x < 1$ and $f'(x) < 0$ for $x > 1$, the critical point corresponds to the maximum value of f. The absolute maximum of f occurs at $x = 1$.

(b) To find the values of b, use grapher techniques to solve $xe^{-x} = 0.1e^{-0.1}$, $xe^{-x} = 0.2e^{-0.2}$, and so on. To find the values of A, calculate $(b-a)ae^{-a}$, using the unrounded values of b. (Use the *list* features of the grapher in order to keep track of the unrounded values for part (d).)

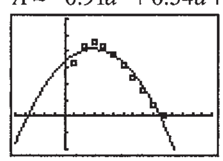
a	b	A
0.1	3.71	0.33
0.2	2.86	0.44
0.3	2.36	0.46
0.4	2.02	0.43
0.5	1.76	0.38
0.6	1.55	0.31
0.7	1.38	0.23
0.8	1.23	0.15
0.9	1.11	0.08
1.0	1.00	0.00

(c) 

[0, 1.1] by [−0.2, 0.6]

(d) Quadratic:

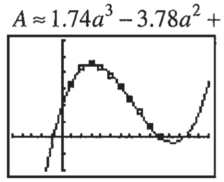
$A \approx -0.91a^2 + 0.54a + 0.34$



[−0.5, 1.5] by [−0.2, 0.6]

Cubic:

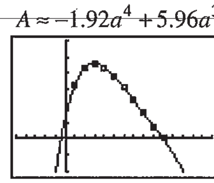
$A \approx 1.74a^3 - 3.78a^2 + 1.86a + 0.19$



[−0.5, 1.5] by [−0.2, 0.6]

Quartic:

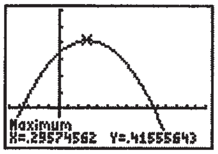
$A \approx -1.92a^4 + 5.96a^3 - 6.87a^2 + 2.71a + 0.12$



[−0.5, 1.5] by [−0.2, 0.6]

28. Continued

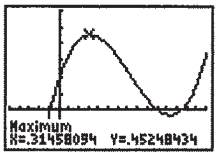
(e) Quadratic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the quadratic regression equation, the maximum area occurs at $a \approx 0.30$ and is approximately 0.42.

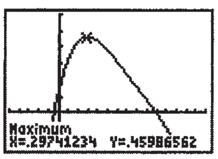
Cubic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the cubic regression equation, the maxiumu area occurs at $a \approx 0.31$ and is approximately 0.45.

Quartic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the quartic regression equation the maximum area occurs at $a \approx 0.30$ and is approximately 0.46.

29. (a) $f'(x)$ is a quadratic polynominal, and as such it can have 0, 1, or 2 zeros. If it has 0 or 1 zeros, then its sign never changes, so $f(x)$ has no local extrema.
If $f'(x)$ has 2 zeros, then its sign changes twice, and $f(x)$ has 2 local extrema at those points.

(b) Possible answers:
No local extrema: $y = x^3$;
2 local extrema: $y = x^3 - 3x$

30. Let x be the length in inches of each edge of the square end, and let y be the length of the box. Then we require $4x + y \le 108$. Since our goal is to maximize volume, we assume $4x + y = 108$ and so $y = 108 - 4x$. The volume is $V(x) = x^2(108 - 4x) = 108x^2 - 4x^3$, where $0 < x < 27$. Then $V' = 216x - 12x^2 = -12x(x - 18)$, so the critical point occurs at $x = 18$ in. Since $V'(x) > 0$ for $0 < x < 18$ and $V'(x) < 0$ for $18 < x < 27$, the critical point corresponds to the maximum volume. The dimensions of the box with the largest possible volume are 18 in. by 18 in. by 36 in.

31. Since $2x + 2y = 36$, we know that $y = 18 - x$. In part (a), the radius is $\dfrac{x}{2\pi}$ and the height is $18 - x$, and so the volume is given by

$$\pi r^2 h = \pi\left(\frac{x}{2\pi}\right)^2 (18 - x) = \frac{1}{4\pi}x^2(18 - x).$$

In part (b), the radius is x and the height is $18 - x$, and so the volume is given by $\pi r^2 h = \pi x^2(18 - x)$. Thus, each problem requires us to find the value of x that maximizes $f(x) = x^2(18 - x)$ in the interval $0 < x < 18$, so the two problems have the same answer.
To solve either problem, note that $f(x) = 18x^2 - x^3$ and so $f'(x) = 36x - 3x^2 = -3x(x - 12)$. The critical point occursat $x = 12$. Since $f'(x) > 0$ for $0 < x < 12$ and $f'(x) < 0$ for $12 < x < 18$, the critical point corresponds to the maximum value of $f(x)$. To maximize the volume in either part (a) or (b), let $x = 12$ cm and $y = 6$ cm.

32. Note that $h^2 + r^2 = 3$ and so $r = \sqrt{3 - h^2}$. Then the volume is given by $V = \dfrac{\pi}{3}r^2 h = \dfrac{\pi}{3}(3 - h^2)h = \pi h - \dfrac{\pi}{3}h^3$ for $0 < h < \sqrt{3}$, and so $\dfrac{dV}{dh} = \pi - \pi h^2 = \pi(1 - h^2)$. The critical point (for $h > 0$) occurs at $h = 1$. Since $\dfrac{dV}{dh} > 0$ for $0 < h < 1$ and $\dfrac{dV}{dh} < 0$ for $1 < h < \sqrt{3}$, the critical point corresponds to the maximum volume. The cone of greatest volume has radius $\sqrt{2}$ m, height 1 m, and volume $\dfrac{2\pi}{3}\text{m}^3$.

33. (a) We require $f(x)$ to have a critical point at $x = 2$. Since $f'(x) = 2x - ax^{-2}$, we have $f'(2) = 4 - \dfrac{a}{4}$ and so our requirement is that $4 - \dfrac{a}{4} = 0$. Therefore, $a = 16$. To verify that the critical point corresponds to a local minimum, note that we now have $f'(x) = 2x - 16x^{-2}$ and so $f''(x) = 2 + 32x^{-3}$, so $f''(2) = 6$, which is positive as expected. So, use $a = 16$.

(b) We require $f''(1) = 0$. Since $f'' = 2 + 2ax^{-3}$, we have $f''(1) = 2 + 2a$, so our requirement is that $2 + 2a = 0$. Therefore, $a = -1$. To verify that $x = 1$ is in fact an inflection point, note that we now have $f''(x) = 2 - 2x^{-3}$, which is negative for $0 < x < 1$ and positive for $x > 1$. Therefore, the graph of f is concave down in the interval (0, 1) and concave up in the interval $(1, \infty)$, So, use $a = -1$.

34. $f'(x) = 2x - ax^{-2} = \frac{2x^3 - a}{x^2}$, so the only sign change in $f'(x)$ occurs at $x = \left(\frac{a}{2}\right)^{1/3}$, where the sign changes from negative to positive. This means there is a local minimum at that point, and there are no local maxima.

35. (a) Note that $f'(x) = 3x^2 + 2ax + b$. We require $f'(-1) = 0$ and $f'(3) = 0$, which give $3 - 2a + b = 0$ and $27 + 6a + b = 0$. Substracting the first equation from the second, we have $24 + 8a = 0$ and so $a = -3$. Substituting into the first equation, we have $9 + b = 0$, so $b = -9$. Therefore, our equation for $f(x)$ is $f(x) = x^3 - 3x^2 - 9x$. To verify that we have a local maximum at $x = -1$ and a local minimum at $x = 3$, note that $f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$, which is positive for $x < -1$, negative for $-1 < x < 3$, and positive for $x > 3$. So, use $a = -3$ and $b = -9$.

(b) Note that $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$. We require $f'(4) = 0$ and $f''(1) = 0$, which give $48 + 8a + b = 0$ and $6 + 2a = 0$. By the second equation, $a = -3$, and so the first equation becomes $48 - 24 + b = 0$. Thus $b = -24$. To verify that we have a local minimum at $x = 4$, and an inflection point *at* $x = 1$, note that we now have $f''(x) = 6x - 6$. Since f'' changes sign at $x = 1$ and is positive at $x = 4$, the desired conditions are satisfied. So, use $a = -3$ and $b = -24$.

36. Refer to the illustration in the problem statement. Since $x^2 + y^2 = 9$, we have $x = \sqrt{9 - y^2}$. Then the volume of the cone is given by

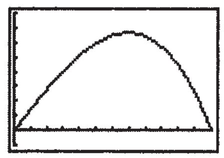
$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi x^2 (y+3)$$
$$= \frac{1}{3}\pi(9 - y^2)(y+3)$$
$$= \frac{\pi}{3}(-y^3 - 3y^2 + 9y + 27),$$

for $-3 < y < 3$.

Thus $\frac{dV}{dy} = \frac{\pi}{3}(-3y^2 - 6y + 9) = -\pi(y^2 + 2y - 3)$
$= -\pi(y+3)(y-1)$, so the critical point in the interval $(-3, 3)$ is $y = 1$. Since $\frac{dV}{dy} > 0$ for $-3 < y < 1$ and $\frac{dV}{dy} < 0$ for $1 < y < 3$, the critical point does correspond to the maximum value, which is $V(1) = \frac{32\pi}{3}$ cubic units.

37. (a) Note that $w^2 + d^2 = 12^2$, so $d = \sqrt{144 - w^2}$. Then we may write $S = kwd^2 = kw(144 - w^2) = 144kw - kw^3$ for $0 < w < 12$, so $\frac{dS}{dw} = 144k - 3kw^2 = -3k(w^2 - 48)$. The critical point (for $0 < w < 12$) occurs at $w = \sqrt{48} = 4\sqrt{3}$. Since $\frac{dS}{dw} > 0$ for $0 < w < 4\sqrt{3}$ and $\frac{dS}{dw} < 0$ for $4\sqrt{3} < w < 12$, the critical point corresponds to the maximum strength. The dimensions are $4\sqrt{3}$ in. wide by $4\sqrt{6}$ in. deep.

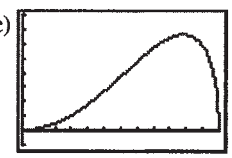
(b)



[0, 12] by [−100, 800]

The graph of $S = 144w - w^3$ is shown. The maximum strength shown in the graph occurs at $w = 4\sqrt{3} \approx 6.9$, which agrees with the answer to part (a).

(c)



[0, 12] by [−100, 800]

The graph of $S = d^2\sqrt{144 - d^2}$ is shown. The maximum strength shown in the graph occurs at $d = 4\sqrt{6} \approx 9.8$, which agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.

Changing the value of k changes the maximum strength, but not the dimensions of the strongest beam. The graphs for different values of k look the same except that the vertical scale is different.

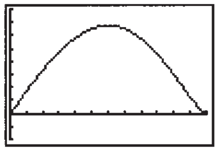
38. (a) Note that $w^2 + d^2 = 12^2$, so $d = \sqrt{144 - w^2}$. Then we may write $S = kwd^3 = kw(144 - w^2)^{3/2}$, so

$$\frac{dS}{dw} = kw \cdot \frac{3}{2}(144 - w^2)^{1/2}(-2w) + k(144 - w^2)^{3/2}$$
$$= (k\sqrt{144 - w^2})(-3w^2 + 144 - w^2)$$
$$= (-4k\sqrt{144 - w^2})(w^2 - 36)$$

The critical point (for $0 < w < 12$) occurs at $w = 6$. Since $\frac{dS}{dw} > 0$ for $0 < w < 6$ and $\frac{dS}{dw} < 0$ for $6 < w < 12$, the critical point corresponds to the maximum stiffness. The dimensions are 6 in. wide by $6\sqrt{3}$ in. deep.

38. Continued

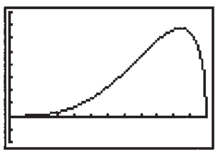
(b)



[0, 12] by [–2000, 8000]

The graph of $S = w(144 - w^2)^{3/2}$ is shown. The maximum stiffness shown in the graph occurs at $w = 6$, which agrees with the answer to part (a).

(c)



[0, 12] by [–2000, 8000]

The graph of $S = d^3\sqrt{144 - d^2}$ is shown. The maximum stiffness shown in the graph occurs at $d = 6\sqrt{3} \approx 10.4$ agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.

Changing the value of k changes the maximum stiffness, but not the dimensions of the stiffest beam. The graphs for different values of k look the same except that the vertical scale is different.

39. (a) $v(t) = s'(t) = -10\pi \sin \pi t$

The speed at time t is $10\pi|\sin \pi t|$. The maximum speed is 10π cm/sec and it occurs at $t = \frac{1}{2}$, $t = \frac{3}{2}$, $t = \frac{5}{2}$, and $t = \frac{7}{2}$ sec. The position at these times is $s = 0$ cm (rest position), and the acceleration $a(t) = v'(t) = -10\pi^2 \cos \pi t$ is 0 cm / sec^2 at these times.

(b) Since $a(t) = -10\pi^2 \cos \pi t$, the greatest magnitude of the acceleration occurs at $t = 0$, $t = 1$, $t = 2$, $t = 3$, and $t = 4$. At these times, the position of the cart is either $s = -10$ cm or $s = 10$ cm, and the speed of the cart is 0 cm/sec.

40. Since $\frac{di}{dt} = -2\sin t + 2\cos t$, the largest magnitude of the current occurs when $-2 \sin t + 2 \cos t = 0$, or $\sin t = \cos t$. Squaring both sides gives $\sin^2 t = \cos^2 t$, and we know that $\sin^2 t + \cos^2 t = 1$, so $\sin^2 t = \cos^2 t = \frac{1}{2}$. Thus the possible values of t are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and so on. Eliminating extraneous solutions, the solutions of $\sin t = \cos t$ are $t = \frac{\pi}{4} + k\pi$ for integers k, and at these times $|i| = |2\cos t + 2\sin t| = 2\sqrt{2}$. The peak current is $2\sqrt{2}$ amps.

41. The square of the distance is

$$D(x) = \left(x - \frac{3}{2}\right)^2 + \left(\sqrt{x} - 0\right)^2 = x^2 - 2x + \frac{9}{4},$$

so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

42. Calculus method:

The square of the distance from the point $(1, \sqrt{3})$ to $(x, \sqrt{16 - x^2})$ is given by

$$\begin{aligned} D(x) &= (x-1)^2 + (\sqrt{16 - x^2} - \sqrt{3})^2 \\ &= x^2 - 2x + 1 + 16 - x^2 - 2\sqrt{48 - 3x^2} + 3 \\ &= -2x + 20 - 2\sqrt{48 - 3x^2}. \end{aligned}$$

Then

$$D'(x) = -2 - \frac{2}{2\sqrt{48 - 3x^2}}(-6x) = -2 + \frac{6x}{\sqrt{48 - 3x^2}}.$$

Solving $D'(x) = 0$, we have:

$$\begin{aligned} 6x &= 2\sqrt{48 - 3x^2} \\ 36x^2 &= 4(48 - 3x^2) \\ 9x^2 &= 48 - 3x^2 \\ 12x^2 &= 48 \\ x &= \pm 2 \end{aligned}$$

We discard $x = -2$ as an extraneous solution, leaving $x = 2$. Since $D'(x) < 0$ for $-4 < x < 2$ and $D'(x) > 0$ for $2 < x < 4$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(2)} = 2$.

Geometry method:

The semicircle is centered at the origin and has radius 4. The distance from the origin to $(1, \sqrt{3})$ is $\sqrt{1^2 + (\sqrt{3})^2} = 2$. The shortest distance from the point to the semicircle is the distance along the radius containing the point $(1, \sqrt{3})$. That distance is $4 - 2 = 2$.

43. No. Since $f(x)$ is a quadratic function and the coefficient of x^2 is positive, it has an absolute minimum at the point where $f'(x)=2x-1=0$, and the point is $\left(\frac{1}{2},\frac{3}{4}\right)$.

44. (a) Because $f(x)$ is periodic with period 2π.

(b) No. Since $f(x)$ is continuous on $[0, 2\pi]$, its absolute minimum occurs at a critical point or endpoint.
Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x)=-4\sin x-2\sin 2x &= 0\\ -4\sin x-4\sin x\cos x &= 0\\ -4(\sin x)(1+\cos x) &= 0\\ \sin x &= 0 \text{ or } \cos x=-1\\ x &= 0, \pi, 2\pi \end{aligned}$$

The critical points (and endpoints) are $(0, 8)$, $(\pi, 0)$, and $(2\pi, 8)$. Thus, $f(x)$ has an absolute minimum at $(\pi, 0)$ and it is never negative.

45. (a)

$$\begin{aligned} 2\sin t &= \sin 2t\\ 2\sin t &= 2\sin t\cos t\\ 2(\sin t)(1-\cos t) &= 0\\ \sin t &= 0 \text{ or } \cos t=1 \end{aligned}$$

$t=k\pi$, where k is an integer

The masses pass each other whenever t is an integer multiple of π seconds.

(b) The vertical distance between the objects is the absolute value of $f(x)=\sin 2t-2\sin t$.
Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x)=2\cos 2t-2\cos t &= 0\\ 2(2\cos^2 t-1)-2\cos t &= 0\\ 2(2\cos^2 t-\cos t-1) &= 0\\ 2(2\cos t+1)(\cos t-1) &= 0\\ \cos t &= -\frac{1}{2} \text{ or } \cos t=1\\ t &= \frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi \end{aligned}$$

The critical points (and endpoints) are $(0, 0)$, $\left(\frac{2\pi}{3},-\frac{3\sqrt{3}}{2}\right)$, $\left(\frac{4\pi}{3},\frac{3\sqrt{3}}{2}\right)$, and $(2\pi, 0)$

The distance is greatest when $t=\frac{2\pi}{3}$ sec and when $t=\frac{4\pi}{3}$ sec. The distance at those times is $\frac{3\sqrt{3}}{2}$ meters.

46. (a)

$$\begin{aligned} \sin t &= \sin\left(t+\frac{\pi}{3}\right)\\ \sin t &= \sin t\cos\frac{\pi}{3}+\cos t\sin\frac{\pi}{3}\\ \sin t &= \frac{1}{2}\sin t+\frac{\sqrt{3}}{2}\cos t\\ \frac{1}{2}\sin t &= \frac{\sqrt{3}}{2}\cos t\\ \tan t &= \sqrt{3} \end{aligned}$$

Solving for t, the particles meet at $t=\frac{\pi}{3}$ sec and at $t=\frac{4\pi}{3}$ sec.

(b) The distance between the particles is the absolute value of $f(t)=\sin\left(t+\frac{\pi}{3}\right)-\sin t=\frac{\sqrt{3}}{2}\cos t-\frac{1}{2}\sin t$. Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(t)=-\frac{\sqrt{3}}{2}\sin t-\frac{1}{2}\cos t &= 0\\ -\frac{\sqrt{3}}{2}\sin t &= \frac{1}{2}\cos t\\ \tan t &= -\frac{1}{\sqrt{3}} \end{aligned}$$

The solutions are $t=\frac{5\pi}{6}$ and $t=\frac{11\pi}{6}$, so the critical points are at $\left(\frac{5\pi}{6},-1\right)$ and $\left(\frac{11\pi}{6},1\right)$, and the interval endpoints are at $\left(0,\frac{\sqrt{3}}{2}\right)$, and $\left(2\pi,\frac{\sqrt{3}}{2}\right)$. The particles are farthest apart at $t=\frac{5\pi}{6}$ sec and at $t=\frac{11\pi}{6}$ sec, and the maximum distance between the particles is 1 m.

(c) We need to maximize $f'(t)$, so we solve $f''(t)=0$.

$$\begin{aligned} f''(t)=-\frac{\sqrt{3}}{2}\cos t+\frac{1}{2}\sin t &= 0\\ \frac{1}{2}\sin t &= \frac{\sqrt{3}}{2}\cos t \end{aligned}$$

This is the same equation we solved in part (a), so the solutions are $t=\frac{\pi}{3}$ sec and $t=\frac{4\pi}{3}$ sec.

For the function $y=f'(t)$, the critical points occur at $\left(\frac{\pi}{3},-1\right)$ and $\left(\frac{4\pi}{3},1\right)$, and the interval endpoints are at $\left(0,-\frac{1}{2}\right)$ and $\left(2\pi,-\frac{1}{2}\right)$.

46. Continued

Thus, $|f'(t)|$ is maximized at $t=\frac{\pi}{3}$ and $t=\frac{4\pi}{3}$. But these are the instants when the particles pass each other, so the graph of $y=|f(t)|$ has corners at these points and $\frac{d}{dt}|f(t)|$ is undefined at these instants. We cannot say that the distance is changing the fastest at any particular instant, but we can say that near $t=\frac{\pi}{3}$ or $t=\frac{4\pi}{3}$ the distance is changing faster than at any other time in the interval.

47. The trapezoid has height $(\cos\theta)$ft and the trapezoid bases measure 1 ft and $(1+2\sin\theta)$ft, so the volume is given by

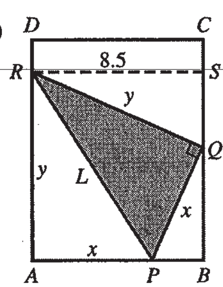
$$V(\theta)=\frac{1}{2}(\cos\theta)(1+1+2\sin\theta)(20)$$
$$=20(\cos\theta)(1+\sin\theta).$$

Find the critical points for $0\le\theta<\frac{\pi}{2}$:

$$V'(\theta)=20(\cos\theta)(\cos\theta)+20(1+\sin\theta)(-\sin\theta)=0$$
$$20\cos^2\theta-20\sin\theta-20\sin^2\theta=0$$
$$20(1-\sin^2\theta)-20\sin\theta-20\sin^2\theta=0$$
$$-20(2\sin^2\theta+\sin\theta-1=0$$
$$-20(2\sin\theta-1)(\sin\theta+1)=0$$
$$\sin\theta=\frac{1}{2}\text{ or }\sin\theta=-1$$
$$\theta=\frac{\pi}{6}$$

The critical point is at $\left(\frac{\pi}{6},15\sqrt{3}\right)$. Since $V'(\theta)>0$ for $0\le\theta<\frac{\pi}{6}$ and $V'(\theta)<0$ for $\frac{\pi}{6}<\theta<\frac{\pi}{2}$, the critical point corresponds to the maximum possible trough volume. The volume is maximized when $\theta=\frac{\pi}{6}$.

48. (a)



Sketch segment RS as shown, and let y be the length of segment QR. Note that $PB = 8.5 - x$, and so

$$QB=\sqrt{x^2-(8.5-x)^2}=\sqrt{8.5(2x-8.5)}.$$

Also note that triangles QRS and PQB are similar.

$$\frac{QR}{RS}=\frac{PQ}{QB}$$
$$\frac{y}{8.5}=\frac{x}{\sqrt{8.5(2x-8.5)}}$$

(a) $$\frac{y^2}{8.5^2}=\frac{x^2}{8.5(2x-8.5)}$$
$$y^2=\frac{8.5x^2}{2x-8.5}$$
$$L^2=x^2+y^2$$
$$L^2=x^2+\frac{8.5x^2}{2x-8.5}$$
$$L^2=\frac{x^2(2x-8.5)+8.5x^2}{2x-8.5}$$
$$L^2=\frac{2x^3}{2x-8.5}$$

(b) Note that $x>4.25$, and let $f(x)=L^2=\frac{2x^3}{2x-8.5}$. Since $y\le 11$, the approximate domain of f is $5.20\le x\le 8.5$. Then

$$f'(x)=\frac{(2x-8.5)(6x^2)-(2x^3)(2)}{(2x-8.5)^2}=\frac{x^2(8x-51)}{(2x-8.5)^2}$$

For $x>5.20$, the critical point occurs at $x=\frac{51}{8}=6.375$ in., and this corresponds to a minimum value of $f(x)$ because $f'(x)<0$ for $5.20<x<6.375$ and $f'(x)>0$ for $x>6.375$. Therefore, the value of x that minimizes L^2 is $x=6.375$ in.

(c) The minimum value of L is $\sqrt{\frac{2(6.375)^3}{2(6.375)-8.5}}\approx 11.04$ in.

49. Since $R=M^2\left(\frac{C}{2}-\frac{M}{3}\right)=\frac{C}{2}M^2-\frac{1}{3}M^3$, we have $\frac{dR}{dM}=CM-M^2$. Let $f(M)=CM-M^2$. Then $f'(M)=C-2M$, and the critical point for f occurs at $M=\frac{C}{2}$. This value corresponds to a maximum because $f'(M)>0$ for $M<\frac{C}{2}$ and $f'(M)<0$ for $M>\frac{C}{2}$. The value of M that maximizes $\frac{dR}{dM}$ is $M=\frac{C}{2}$.

50. The profit is given by

$$P(x) = (n)(x-c) = a + b(100-x)(x-c)$$
$$= -bx^2 + (100+c)bx + (a - 100bc).$$

Then $P'(x) = -2bx + (100+c)b$
$$= b(100 + c - 2x).$$

The critical point occurs at $x = \frac{100+c}{2} = 50 + \frac{c}{2}$, and this value corresponds to the maximum profit because

$P'(x) > 0$ for $x < 50 + \frac{c}{2}$ and $P'(x) < 0$ for $x > 50 + \frac{c}{2}$.

A selling price of $50 + \frac{c}{2}$ will bring the maximum profit.

51. True. This is guaranteed by the Extreme Value Theorem (Section 4.1).

52. False. For example, consider $f(x) = x^3$ at $c = 0$.

53. D. $f(x) = x^2(60-x)$

$$f'(x) = x^2(-1) + (60-x)(2x)$$
$$= -x^2 + 120x - 2x^2$$
$$= -3x^2 + 120x$$
$$= -3x(x-40)$$

$x = 0$ or $x = 40$

$60 - x = 60$ $\quad$ $60 - x = 20$

$x^2(60-x) = 0$

$(40)^2(20) = (1600)(20)$
$= 32{,}000$

54. B. Since $f'(x)$ is negative, $f(x)$ is always decreasing, so $f(25) = 3$.

55. B. $A = \frac{1}{2}bh$

$$b^2 + h^2 = 100$$
$$b = \sqrt{100 - h^2}$$
$$A = \frac{h}{2}\sqrt{100 - h^2}$$
$$A' = \frac{\sqrt{100-h^2}}{2} - \frac{h^2}{2\sqrt{100-h^2}}$$

$A' = 0$ when $h = \sqrt{50}$

$$b = \sqrt{100 - \sqrt{50}^2} = \sqrt{50}$$
$$A_{\text{max}} = \frac{1}{2}\sqrt{50}\sqrt{50} = 25$$

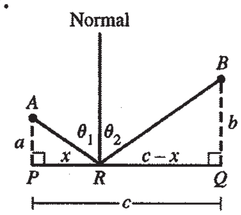
56. E. length $= 2x$

height $= 30 - x^2 - 4x^2 = 30 - 5x^2$

area $= A = 2x(30 - 5x^2) = 60x - 10x^3$

$$\frac{dA}{dx}(60x - 10x^3) = 60 - 30x^2$$
$$x = \sqrt{2}$$
$$2\sqrt{2}\left(30 - 5(\sqrt{2})^2\right) = 40\sqrt{2}.$$

57.



Let P be the foot of the perpendicular from A to the mirror, and Q be the foot of the perpendicular from B to the mirror. Suppose the light strikes the mirror at point R on the way from A to B. Let:

a = distance from A to P
b = distance from B to Q
c = distance from P to Q
x = distance from P to R

To minimize the time is to minimize the total distance the light travels going from A to B. The total distance is

$$D(x) = \sqrt{x^2 + a^2} + \sqrt{(c-x)^2 + b^2}$$

Then

$$D'(x) = \frac{1}{2\sqrt{x^2+a^2}}(2x) + \frac{1}{2\sqrt{(c-x)^2+b^2}}[-2(c-x)]$$
$$= \frac{x}{\sqrt{x^2+a^2}} - \frac{c-x}{\sqrt{(c-x)^2+b^2}}$$

Solving $D'(x) = 0$ gives the equation

$\frac{x}{\sqrt{x^2+a^2}} = \frac{c-x}{\sqrt{(c-x)^2+b^2}}$ which we will refer to as Equation 1. Squaring both sides, we have:

$$\frac{x^2}{x^2+a^2} = \frac{(c-x)^2}{(c-x)^2+b^2}$$
$$x^2\left[(c-x)^2 + b^2\right] = (c-x)^2(x^2+a^2)$$
$$x^2(c-x)^2 + x^2b^2 = (c-x)^2x^2 + (c-x)^2a^2$$
$$x^2b^2 = (c-x)^2a^2$$
$$x^2b^2 = \left[c^2 - 2xc + x^2\right]a^2$$
$$0 = (a^2 - b^2)x^2 - 2a^2cx + a^2c^2$$
$$0 = [(a+b)x - ac][(a-b)x - ac]$$
$$x = \frac{ac}{a+b} \text{ or } x = \frac{ac}{a-b}$$

Note that the value $x = \frac{ac}{a-b}$ is an extraneous solution because $c - x = c - \frac{ac}{a-b} = \frac{-cb}{a-b}$, so x and $c - x$ could not both be positive. The only critical point occurs at $x = \frac{ac}{a+b}$.

57. Continued

To verify that critical point represents the minimum distance, notice that D is differentiable for all x in $[0, c]$ with a single critical point in the interior of the inverval. Since $D'(0) = \frac{-c}{\sqrt{c^2+b^2}} < 0$, D must be decreasing from 0 to the critical point, and since $D'(c) = \frac{c}{\sqrt{c^2+b^2}} > 0$, D must be increasing from the critical point to c.

We now know that $D(x)$ is minimized when Equation 1 is true, or, equivalently, $\frac{PR}{AR} = \frac{QR}{BR}$. This means that the two right triangles APR and BQR are similar, which in turn implies that the two angles must be equal.

58. $\frac{dv}{dx} = ka - 2kx$

The critical point occurs at $x = \frac{ka}{2k} = \frac{a}{2}$, which represents a maximum value because $\frac{d^2v}{dx^2} = -2k$, which is negative for all x. The maximum value of v is

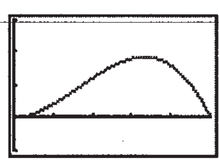
$$kax - kx^2 = ka\left(\frac{a}{2}\right) - k\left(\frac{a}{2}\right)^2 = \frac{ka^2}{4}.$$

59. (a) $v = cr_0r^2 - cr^3$

$$\frac{dv}{dr} = 2cr_0r - 3cr^2 = cr(2r_0 - 3r)$$

The critical point occurs at $r = \frac{2r_0}{3}$. (Note that $r = 0$ is not in the domain of v.) The critical point represents a maximum because $\frac{d^2v}{dr^2} = 2cr_0 - 6cr = 2c(r_0 - 3r)$, which is negative when $r = \frac{2r_0}{3}$.

(b) We graph $v = (0.5 - r)r^2$, and observe that the maximum indeed occurs at $v = \left(\frac{2}{3}\right)0.5 = \frac{1}{3}$.



[0, 0.5] by [−0.01, 0.03]

60. (a) Since $A'(q) = -kmq^{-2} + \frac{h}{2}$, the critical point occurs when $\frac{km}{q^2} = \frac{h}{2}$, or $q = \sqrt{\frac{2km}{h}}$. This corresponds to the minimum value of $A(q)$ because $A''(q) = 2kmq^{-3}$, which is positive for $q > 0$.

(b) The new formula for average weekly cost is

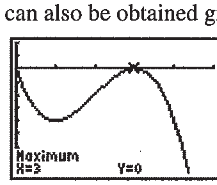
$$\begin{aligned} B(q) &= \frac{(k+bq)m}{q} + cm + \frac{hq}{2} \\ &= \frac{km}{q} + bm + cm + \frac{hq}{2} \\ &= A(q) + bm \end{aligned}$$

Since $B(q)$ differs from $A(q)$ by a constant, the minimum value of $B(q)$ will occur at the same q-value as the minimum value of $A(q)$. The most economical quantity is again $\sqrt{\frac{2km}{h}}$.

61. The profit is given by

$$\begin{aligned} p(x) &= r(x) - c(x) \\ &= 6x - (x^3 - 6x^2 + 15x) \\ &= -x^3 + 6x^2 - 9x, \text{ for } x \geq 0. \end{aligned}$$

Then $p'(x) = -3x^2 + 12x - 9 = -3(x-1)(x-3)$, so the critical points occur at $x = 1$ and $x = 3$. Since $p'(x) < 0$ for $0 \leq x < 1$, $p'(x) > 0$ for $1 < x < 3$, and $p'(x) < 0$ for $x > 3$, the relative maxima occur at the endpoint $x = 0$ and at the critical point $x = 3$. Since $p(0) = p(3) = 0$, this means that for $x \geq 0$, the function $p(x)$ has its absolute maximum value at the points (0, 0) and (3, 0). This result can also be obtained graphically, as shown.



[0, 5] by [−8, 2]

62. The average cost is given by $a(x) = \frac{c(x)}{x} = x^2 - 20x + 20{,}000$. Therefore, $a'(x) = 2x - 20$ and the critical value is $x = 10$, which represents the minimum because $a''(x) = 2$, which is positive for all x. The average cost is minimized at a production level of 10 items.

63. **(a)** According to the graph, $y'(0) = 0$.

(b) According to the graph, $y'(-L) = 0$.

(c) $y(0) = 0$, so $d = 0$.

Now $y'(x) = 3ax^2 + 2bx + c$, so $y'(0) = 0$ implies that $c = 0$. Therefore, $y(x) = ax^3 + bx^2$ and $y'(x) = 3ax^2 + 2bx$. Then $y(-L) = -aL^3 + bL^2 = H$ and $y'(-L) = 3aL^2 - 2bL = 0$, so we have two linear equations in the two unknowns a and b. The second equation gives $b = \frac{3aL}{2}$. Substituting into the first equation, we have $-aL^3 + \frac{3aL^3}{2} = H$, or $\frac{aL^3}{2} = H$, so $a = 2\frac{H}{L^3}$. Therefore, $b = 3\frac{H}{L^2}$ and the equation for y is $y(x) = 2\frac{H}{L^3}x^3 + 3\frac{H}{L^2}x^2$, or

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$

64. **(a)** The base radius of the cone is $r = \frac{2\pi a - x}{2\pi}$ and so the height is $h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \left(\frac{2\pi a - x}{2\pi}\right)^2}$. Therefore,

$$V(x) = \frac{\pi}{3}r^2 h = \frac{\pi}{3}\left(\frac{2\pi a - x}{2\pi}\right)^2 \sqrt{a^2 - \left(\frac{2\pi a - x}{2\pi}\right)^2}.$$

(b) To simplify the calculations, we shall consider the volume as a function of r:

volume $= f(r) = \frac{\pi}{3}r^2\sqrt{a^2 - r^2}$, where $0 < r < a$.

$$f'(r) = \frac{\pi}{3}\frac{d}{dr}(r^2\sqrt{a^2 - r^2})$$
$$= \frac{\pi}{3}\left[r^2 \cdot \frac{1}{2\sqrt{a^2 - r^2}} \cdot (-2r) + (\sqrt{a^2 - r^2})(2r)\right]$$
$$= \frac{\pi}{3}\left[\frac{-r^3 + 2r(a^2 - r^2)}{\sqrt{a^2 - r^2}}\right] = \frac{\pi}{3}\left[\frac{(2a^2 r - 3r^3)}{\sqrt{a^2 - r^2}}\right]$$
$$= \frac{\pi r(2a^2 - 3r^2)}{3\sqrt{a^2 - r^2}}$$

The critical point occurs when $r^2 = \frac{2a^2}{3}$, which gives $r = a\sqrt{\frac{2}{3}} = \frac{a\sqrt{6}}{3}$. Then $h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \frac{2a^2}{3}} = \sqrt{\frac{a^2}{3}} = \frac{a\sqrt{3}}{3}$. Using $r = \frac{a\sqrt{6}}{3}$ and $h = \frac{a\sqrt{3}}{3}$, we may now find the values of r and h for the given values of a

when $a = 4: r = \frac{4\sqrt{6}}{3}, h = \frac{4\sqrt{3}}{3}$;

when $a = 5: r = \frac{5\sqrt{6}}{3}, h = \frac{5\sqrt{3}}{3}$;

when $a = 6: r = 2\sqrt{6}, h = 2\sqrt{3}$;

when $a = 8: r = \frac{8\sqrt{6}}{3}, h = \frac{8\sqrt{3}}{3}$

(c) Since $r = \frac{a\sqrt{6}}{3}$ and $h = \frac{a\sqrt{3}}{3}$, the relationship is $\frac{r}{h} = \sqrt{2}$.

65. **(a)** Let x_0 represent the fixed value of x at point P, so that P has coordinates (x_0, a) and let $m = f'(x_0)$ be the slope of line RT. Then the equation of line RT is $y = m(x - x_0) + a$. The y-intercept of this line is $m(0 - x_0) + a = a - mx_0$, and the x-intercept is the solution of $m(x - x_0) + a = 0$, or $x = \frac{mx_0 - a}{m}$. Let O designate the origin. Then (Area of triangle RST)

$= 2$ (Area of triangle ORT)

$= 2 \cdot \frac{1}{2}$ (x-intercept of line RT) (y-intercept of line RT)

$$= 2 \cdot \frac{1}{2}\left(\frac{mx_0 - a}{m}\right)(a - mx_0)$$
$$= -m\left(\frac{mx_0 - a}{m}\right)\left(\frac{mx_0 - a}{m}\right)$$
$$= -m\left(\frac{mx_0 - a}{m}\right)^2$$
$$= -m\left(x_0 - \frac{a}{m}\right)^2$$

Substituting x for x_0, $f'(x)$ for m, and $f(x)$ for a, we have $A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$.

65. Continued

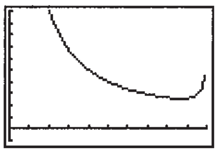
(b) The domain is the open interval (0, 10).

To graph, let $y_1 = f(x) = 5 + 5\sqrt{1 - \dfrac{x^2}{100}}$,

$y_2 = f'(x) = \text{NDER}(y_1)$, and

$$y_3 = A(x) = -y_2\left(x - \frac{y_1}{y^2}\right)^2.$$

The graph of the area function $y_3 = A(x)$ is shown below.



[0, 10] by [–100, 1000]

The vertical asymptotes at $x = 0$ and $x = 10$ correspond to horizontal or vertical tangent lines, which do not form triangles.

(c) Using our expression for the y-intercept of the tangent line, the height of the triangle is

$$a - mx = f(x) - f'(x) \cdot x$$
$$= 5 + \frac{1}{2}\sqrt{100 - x^2} - \frac{-x}{2\sqrt{100 - x^2}}x$$
$$= 5 + \frac{1}{2}\sqrt{100 - x^2} + \frac{x^2}{2\sqrt{100 - x^2}}$$

We may use graphing methods or the analytic method in part (d) to find that the minimum value of $A(x)$ occurs at $x \approx 8.66$. Substituting this value into the expression above, the height of the triangle is 15. This is 3 times the y-coordinate of the center of the ellipse.

(d) Part (a) remains unchanged. The domain is $(0, C)$. To graph, note that

$$f(x) = B + B\sqrt{1 - \frac{x^2}{C^2}} = B + \frac{B}{C}\sqrt{C^2 - x^2} \quad \text{and}$$

$$f'(x) = \frac{B}{C}\frac{1}{2\sqrt{C^2 - x^2}}(-2x) = \frac{-Bx}{C\sqrt{C^2 - x^2}}.$$

Therefore, we have

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{B + \frac{B}{C}\sqrt{C^2 - x^2}}{\frac{-Bx}{C\sqrt{C^2 - x^2}}}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{(BC + B\sqrt{C^2 - x^2})\sqrt{C^2 - x^2}}{-Bx}\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + (BC + B\sqrt{C^2 - x^2})(\sqrt{C^2 - x^2})\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + BC\sqrt{C^2 - x^2} + B(C^2 - x^2)\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[BC(C + \sqrt{C^2 - x^2})\right]^2$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})^2}{x\sqrt{C^2 - x^2}}$$

$$A'(x) = BC \cdot \frac{\left(x\sqrt{C^2 - x^2}\right)(2)(C + \sqrt{C^2 - x^2})\left(\dfrac{-x}{\sqrt{C^2 - x^2}}\right) - (C + \sqrt{C^2 - x^2})^2\left(x\dfrac{-x}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}(1)\right)}{x^2(C^2 - x^2)}$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)}\left[-2x^2 - (C + \sqrt{C^2 - x^2})\left(\frac{-x^2}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}\right)\right]$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2\sqrt{C^2 - x^2}}\left[-2x^2 + \frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} + x^2 - (C^2 - x^2)\right]$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)}\left(\frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} - C^2\right)$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)^{3/2}}\left[Cx^2 - C(C^2 - x^2) - C^2\sqrt{C^2 - x^2}\right]$$

$$= \frac{BC^2(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)^{3/2}}(2x^2 - C^2 - C\sqrt{C^2 - x^2})$$

To find the critical points for $0 < x < C$, we solve:

$$2x^2 - C^2 = C\sqrt{C^2 - x^2}$$
$$4x^4 - 4C^2x^2 + C^4 = C^4 - C^2x^2$$
$$4x^4 - 3C^2x^2 = 0$$
$$x^2(4x^2 - 3C^2) = 0$$

65. Continued

The minimum value of $A(x)$ for $0 < x < C$ occurs at the critical point $x = \frac{C\sqrt{3}}{2}$, or $x^2 = \frac{3C^2}{4}$. The corresponding triangle height is

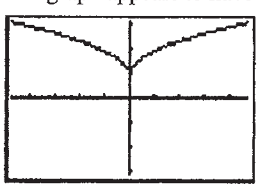
$$\begin{aligned} a - mx &= f(x) - f'(x) \cdot x \\ &= B + \frac{B}{C}\sqrt{C^2 - x^2} + \frac{Bx^2}{C\sqrt{C^2 - x^2}} \\ &= B + \frac{B}{C}\sqrt{C^2 - \frac{3C^2}{4}} + \frac{B\left(\frac{3C^2}{4}\right)}{C\sqrt{C^2 - \frac{3C^2}{4}}} \\ &= B + \frac{B}{C}\left(\frac{C}{2}\right) + \frac{\frac{3BC^2}{4}}{\frac{C^2}{2}} \\ &= B + \frac{B}{2} + \frac{3B}{2} \\ &= 3B \end{aligned}$$

This shows that the triangle has minimum area when its height is $3B$.

Section 4.5 Linerization and Newton's Method (pp. 233–245)

Exploration 1 Appreciating Local Linearity

1. The graph appears to have either a cusp or a corner at (0, 1).



$y = \left(x^2 + 0.0001\right)^{1/4} + 0.9$

2. $f'(x) = \frac{1}{4}\left(x^2 + 0.0001\right)^{-3/4}(2x) = \frac{x}{\sqrt[4]{\left(x^2 + 0.0001\right)^3}}$.

   Since $f'(0) = 0$, the tangent line at (0, 1) has equation $y = 1$.

3. The "corner" becomes smooth and the graph straightens out.

4. As with any differentiable curve, the graph comes to resemble the tangent line.

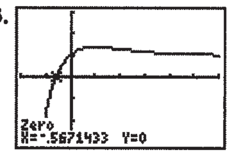
Exploration 2 Using Newton's Method on Your Calculator

See text page 237.

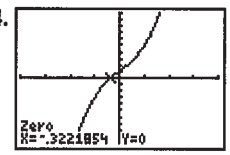
Quick Review 4.5

1. $\frac{dy}{dx} = \cos(x^2 + 1) \cdot \frac{d}{dx}(x^2 + 1)$
   $= 2x\cos(x^2 + 1)$

2. $\frac{dy}{dx} = \frac{(x+1)(1 - \sin x) - (x + \cos x)(1)}{(x+1)^2}$
   $= \frac{x - x\sin x + 1 - \sin x - x - \cos x}{(x+1)^2}$
   $= \frac{1 - \cos x - (x+1)\sin x}{(x+1)^2}$

3. 

   $[-2, 6]$ by $[-3, 3]$
   $x \approx -0.567$

4. 

   $[-4, 4]$ by $[-10, 10]$
   $x \approx -0.322$

5. $f'(x) = (x)(-e^{-x}) + (e^{-x})(1) = e^{-x} - xe^{-x}$
   $f'(0) = 1$
   The lines passes through (0, 1) and has slope 1. Its equation is $y = x + 1$.

6. $f'(x) = (x)(-e^{-x}) + (e^{-x})(1) = e^{-x} - xe^{-x}$
   $f'(-1) = e^1 - (-e^1) = 2e$
   The lines passes through $(-1, -e + 1)$ and has slope $2e$. Its equation is $y = 2e(x+1) + (-e+1)$, or $y = 2ex + e + 1$.

7. **(a)** $x + 1 = 0$
   $x = -1$

   (b) $2ex + e + 1 = 0$
   $2ex = -(e+1)$
   $x = -\frac{e+1}{2e}$
   ≈ -0.684