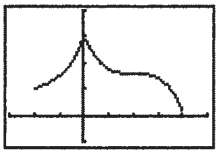
Exploration 2 Finding f from f' and f''

1. f has an absolute maximum at $x = 0$ and an absolute minimum of 1 at $x = 4$. We are not given enough information to determine $f(0)$.

2. f has a point of inflection at $x = 2$.

3. 

[−3, 5] by [−5, 20]

Quick Review 4.3

1. $x^2 - 9 < 0$
$(x+3)(x-3) < 0$

Intervals	$x < -3$	$-3 < x < 3$	$3 < x$
Sign of $(x+3)(x-3)$	+	−	+

Solution set: $(-3, 3)$

2. $x^3 - 4x > 0$
$x(x+2)(x-2) > 0$

Intervals	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x$
Sign of $x(x+2)(x-2)$	−	+	−	+

Solution set: $(-2, 0) \cup (2, \infty)$

3. f: all reals
f': all reals, since $f'(x) = xe^x + e^x$

4. f: all reals
f': $x \neq 0$, since $f'(x) = \frac{3}{5}x^{-2/5}$

5. f: $x \neq 2$
f': $x \neq 2$, since $f'(x) = \frac{(x-2)(1)-(x)(1)}{(x-2)^2} = \frac{-2}{(x-2)^2}$

6. f: all reals
f': $x \neq 0$, since $f'(x) = \frac{2}{5}x^{-3/5}$

7. Left end behavior model: 0
Right end behavior model: $-x^2e^x$
Horizontal asymptote: $y = 0$

8. Left end behavior model: x^2e^{-x}
Right end behavior model: 0
Horizontal asymptote: $y = 0$

9. Left end behavior model: 0
Right end behavior model: 200
Horizontal asymptote: $y = 0$, $y = 200$

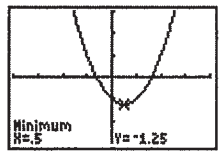
10. Left end behavior model: 0
Right end behavior model: 375
Horizontal asymptotes: $y = 0$, $y = 375$

Section 4.3 Exercises

1. $y' = 2x - 1$

Intervals	$x < \frac{1}{2}$	$x > \frac{1}{2}$
Sign of y'	−	+
Behavior of y	Decreasing	Increasing

Graphical support:



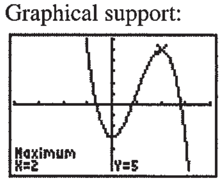
[−4, 4] by [−3, 3]

Local (and absolute) minimum at $\left(\frac{1}{2}, -\frac{5}{4}\right)$

2. $y' = -6x^2 + 12x = -6x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y'	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

Graphical support:



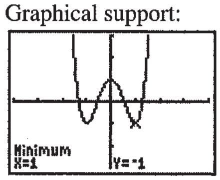
[−4, 4] by [−6, 6]

Local maximum: $(2, 5)$;
local minimum: $(0, -3)$

3. $y' = 8x^3 - 8x = 8x(x-1)(x+1)$

Intervals	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	−	+	−	+
Behavior of y	Decreasing	Increasing	Decreasing	Increasing

Graphical support:



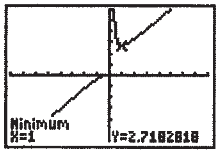
[−4, 4] by [−3, 3]

Local maximum: $(0, 1)$;
local (and absolute) minima: $(-1, -1)$ and $(1, -1)$

4. $y' = xe^{1/x}(-x^{-2}) + e^{1/x} = e^{1/x}\left(1 - \frac{1}{x}\right)$

Intervals	$x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	+	−	+
Behavior of y	Increasing	Decreasing	Increasing

Graphical support:



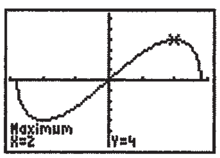
$[-8, 8]$ by $[-6, 6]$

Local minimum: $(1, e)$

5. $y' = x\frac{1}{2\sqrt{8-x^2}}(-2x) + \sqrt{8-x^2}(1) = \frac{8-2x^2}{\sqrt{8-x^2}}$

Intervals	$-\sqrt{8} < x < -2$	$-2 < x < 2$	$2 < x < \sqrt{8}$
Sign of y'	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

Graphical support:



$[-3.02, 3.02]$ by $[-6.5, 6.5]$

Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$;

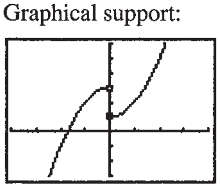
local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$

Note that the local extrema at $x = \pm 2$ are also absolute extrema.

6. $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y'	+	+
Behavior of y	Increasing	Increasing

Graphical support:



$[-4, 4]$ by $[-3, 6]$

Local minimum: $(0, 1)$

7. $y' = 12x^2 + 42x + 36 = 6(x+2)(2x+3)$

$y'' = 24x + 42 = 6(4x+7)$

Intervals	$x < -\frac{7}{4}$	$-\frac{7}{4} < x$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

(a) $\left(-\frac{7}{4}, \infty\right)$

(b) $\left(-\infty, -\frac{7}{4}\right)$

8. $y' = -4x^3 + 12x^2 - 4$

$y'' = -12x^2 + 24x = -12x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y''	−	+	−
Behavior of y	Concave down	Concave up	Concave down

(a) $(0, 2)$

(b) $(-\infty, 0)$ and $(2, \infty)$

9. $y' = \frac{2}{5}x^{-4/5}$

$y'' = -\frac{8}{25}x^{-9/5}$

Intervals	$x < 0$	$0 < x$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

(a) $(-\infty, 0)$

(b) $(0, \infty)$

10. $y' = -\frac{1}{3}x^{-2/3}$

$y'' = \frac{2}{9}x^{-5/3}$

Intervals	$x < 0$	$0 < x$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

(a) $(0, \infty)$

(b) $(-\infty, 0)$

11. $y' = \begin{cases} 2, & x < 1 \\ -2x, & x > 1 \end{cases}$

$y'' = \begin{cases} 0, & x < 1 \\ -2, & x > 1 \end{cases}$

Intervals	$x < 1$	$1 < x$
Sign of y''	0	$-$
Behavior of y	Linear	Concave down

(a) None

(b) $(1, \infty)$

12. $y' = e^x$

$y'' = e^x$

Since y' and y'' are both positive on the entire domain, y is increasing and concave up on the entire domain.

(a) $(0, 2\pi)$

(b) None

13. $y = xe^x$

$y' = e^x + xe^x$

$y'' = 2e^x + xe^x$

$y'' = 0$ at $x = -2$

Intervals	$x < -2$	$x > -2$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

Inflection pt at $\left(-2, -\dfrac{2}{e^2}\right)$

14. $y = x\sqrt{9 - x^2}$

$$y' = \sqrt{9 - x^2} - \frac{x^2}{\sqrt{9 - x^2}}$$

$$y'' = -\frac{x}{(9 - x^2)^{1/2}} + \frac{x^3 - 18x}{(9 - x^2)^{3/2}} = \frac{x(2x^2 - 27)}{(9 - x^2)^{3/2}}.$$

On the domain $[-3, 3]$, $y'' = 0$ only at $x = 0$

Intervals	$-3 < x < 0$	$0 < x < 3$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

Inflection pt at (0, 0)

15. $y' = \dfrac{1}{1 + x^2}$

$$y'' = \frac{d}{dx}(1 + x^2)^{-1} = -(1 + x^2)^{-2}(2x) = \frac{-2x}{(1 + x^2)^2}$$

Intervals	$x < 0$	$0 < x$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

Inflection pt at (0, 0)

16. $y = x^3(4 - x)$

$y' = 12x^2 - 4x^3$

$y'' = 24x - 12x^2$

Intervals	$x < 0$	$0 < x < 2$	$x > 2$
Sign of y''	$-$	$+$	$-$
Behavior of y	Concave down	Concave up	Concave down

Inflection pts at (0, 0) and (2, 16)

17. $y = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}$

$$y' = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4x - 4}{3x^{2/3}}$$

$$y'' = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4x + 8}{9x^{5/3}}$$

Intervals	$x < -2$	$-2 < x < 0$	$0 < x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

Inflection pts at $(-2, 6\sqrt[3]{2}) \approx (-2, 7.56)$ and $(0, 0)$

18. $y = x^{1/2}(x + 3)$

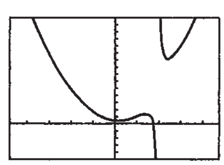
$$y' = \frac{1}{2}x^{-1/2}(x + 3) + x^{1/2}$$

$$y'' = \frac{1}{(x)^{1/2}} - \frac{x + 3}{4(x)^{3/2}} = 0$$

Intervals	$0 < x < 1$	$x > 1$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

Inflection pt at (1, 4)

19. We use a combination of analytic and grapher techniques to solve this problem. Depending on the viewing window chosen, graphs obtained using NDER may exhibit strange behavior near $x = 2$ because, for example, NDER $(y, 2) \approx 1{,}000{,}000$ while y' is actually undefined at $x = 2$. The graph of $y = \dfrac{x^3 - 2x^2 + x - 1}{x - 2}$ is shown below.



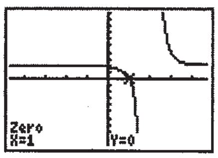
[−4,7, 4.7] by [−5, 15]

19. Continued

$$y' = \frac{(x-2)(3x^2-4x+1)-(x^3-2x^2+x-1)(1)}{(x-2)^2}$$
$$= \frac{2x^3-8x^2+8x-1}{(x-2)^2}$$

$$y'' = \frac{(x-2)^2(6x^2-16x+8)-(2x^3-8x^2+8x-1)(2)(x-2)}{(x-2)^4}$$
$$= \frac{(x-2)(6x^2-16x+8)-2(2x^3-8x^2+8x-1)}{(x-2)^3}$$
$$= \frac{2x^3-12x^2+24x-14}{(x-2)^3}$$
$$= \frac{2(x-1)(x^2-5x+7)}{(x-2)^3}$$

The graph of y'' is shown below.



[−4,7, 4.7] by [−10, 10]

Note that the discriminant of x^2-5x+7 is $(-5)^2-4(1)(7)=-3$, so the only solution of $y''=0$ is $x=1$.

Intervals	$x<1$	$1<x<2$	$2<x$
Sign of y''	+	−	+
Behavior of y	Concave up	Concave down	Concave up

Inflection pt at (1, 1)

20. $y' = \frac{(x^2+1)(1)-x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$

$$y'' = \frac{(x^2+1)^2(-2x)-(-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4}$$
$$= \frac{(x^2+1)(-2x)-4x(-x^2+1)}{(x^2+1)^3}$$
$$= \frac{2x^3-6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

Intervals	$x<-\sqrt{3}$	$-\sqrt{3}<x<0$	$0<x<\sqrt{3}$	$\sqrt{3}<x$
Sign of y''	−	+	−	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

Inflection pts at (0, 0), $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$, and $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$

21. (a) Zero: $x=\pm1$;
positive: $(-\infty, -1)$ and $(1, \infty)$;
negative: $(-1, 1)$

(b) Zero: $x=0$;
positive: $(0, \infty)$;
negative: $(-\infty, 0)$

22. (a) Zero: $x\approx 0, \pm 1.25$;
positive: $(-1.25, 0)$ and $(1.25, \infty)$;
negative: $(-\infty, -1.25)$ and $(0, 1.25)$

(b) Zero: $x\approx \pm 0.7$;
positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
negative: $(-0.7, 0.7)$

23. (a) $(-\infty, -2]$ and $[0, 2]$

(b) $[-2, 0]$ and $[2, \infty)$

(c) Local maxima: $x=-2$ and $x=2$;
local minimum: $x=0$

24. (a) $[-2, 2]$

(b) $(-\infty, -2]$ and $[2, \infty)$

(c) Local maximum: $x=2$;
local minimum: $x=-2$

25. (a) $v(t)=x'(t)=2t-4$

(b) $a(t)=v'(t)=2$

(c) It begins at position 3 moving in a negative direction. It moves to position −1 when $t=2$, and then changes direction, moving in a positive direction thereafter.

26. (a) $v(t)=x'(t)=-2-2t$

(b) $a(t)=v'(t)=-2$

(c) It begins at position 6 and moves in the negative direction thereafter.

27. (a) $v(t)=x'(t)=3t^2-3$

(b) $a(t)=v'(t)=6t$

(c) It begins at position 3 moving in a negative direction. It moves to position 1 when $t=1$, and then changes direction, moving in a positive direction thereafter.

28. (a) $v(t)=x'(t)=6t-6t^2$

(b) $a(t)=v'(t)=6-12t$

(c) It begins at position 0. It starts moving in the positive direction until it reaches position 1 when $t=1$, and then it changes direction. It moves in the negative direction thereafter.

29. **(a)** The velocity is zero when the tangent line is horizontal, at approximately $t = 2.2$, $t = 6$ and $t = 9.8$.

(b) The acceleration is zero at the inflection points, approximately $t = 4$, $t = 8$ and $t = 11$.

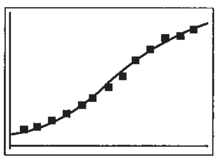
30. **(a)** The velocity is zero when the tangent line is horizontal, at approximately $t = -0.2$, $t = 4$, and $t = 12$.

(b) The acceleration is zero at the inflection points, approximately $t = 1.5$, $t = 5.2$, $t = 8$, $t = 11$, and $t = 13$.

31. Some calculators use different logistic regression equations, so answers may vary.

(a) $y = \dfrac{12655.179}{1+12.871e^{-0.0326t}}$

(b)



[0, 140] by [–200, 12000]

(c) $y = \dfrac{12655.179}{1+12.871e^{-0.0326(180)}} = 12{,}209{,}870$. (This is remarkably close to the 2000 census number of 12,281,054.)

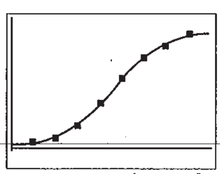
(d) The second derivative has a zero at about 78, indicating that the population was growing fastest in 1898. This corresponds to the inflection point on the regression curve.

(e) The regression equation predicts a population limit of about 12,655,179.

32. Some calculators use different logistic regression equations, so answers may vary.

(a) $y = \dfrac{28984386.288}{1+49.252e^{-0.851t}}$

(b)



[0, 9] by [-3.1×10^6, 3.2×10^7]

(c) The zero of the second derivative is about 4.6, which puts the fastest growth during 1981. This corresponds to the inflection point on the regression curve.

(d) The regression curve predicts that cable subscribers will approach a limit of 28,984,386 + 12,168,450 subscribers (about 41 million).

33. $y = 3x - x^3 + 5$

$y' = 3 - 3x^2$
$y'' = -6x$
$y' = 0$ at ± 1.

$y''(-1) > 0$ and $y''(1) < 0$, so there is a local minimum at $(-1, 3)$ and a local maximum at $(1, 7)$.

34. $y = x^5 - 80x + 100$

$y' = 5x^4 - 80$
$y'' = 20x^3$
$y' = 0$ at ± 2

$y''(-2) < 0$ and $y''(2) > 0$, so there is a local maximum at $(-2, 228)$ and a local minimum at $(2, -28)$.

35. $y = x^3 + 3x^2 - 2$

$y' = 3x^2 + 6x$
$y'' = 6x + 6$
$y' = 0$ at -2 and 0.
$y''(-2) < 0$, $y''(0) > 0$,

so there is a local maximum at $(-2, 2)$ and a local minimum at $(0, -2)$.

36. $y = 3x^5 - 25x^3 + 60x + 20$

$y' = 15x^4 - 75x^2 + 60$
$y'' = 60x^3 - 150x$
$y' = 0$ at ± 1 and ± 2.
$y''(-2) < 0$, $y''(-1) > 0$
$y''(1) < 0$, and $y''(2) > 0$;

so there are local maxima at $(-2, 4)$ and $(1, 58)$, and there are local minima at $(-1, -18)$ and $(2, 36)$.

37. $y = xe^x$

$y' = (x+1)e^x$
$y'' = (x+2)e^x$
$y' = 0$ at -1.
$y''(-1) > 0$, so there is a local minimum at $(-1, -1/e)$.

38. $y = xe^{-x}$

$y' = (1-x)e^{-x}$
$y'' = (x-2)e^{-x}$
$y' = 0$ at 1
$y''(1) < 0$, so there is a local maximum at $(1, 1/e)$.

39. $y' = (x-1)^2(x-2)$

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y'	–	–	+
Behavior of y	Decreasing	Decreasing	Increasing

39. Continued

$$\begin{aligned} y'' &= (x-1)^2(1)+(x-2)(2)(x-1) \\ &= (x-1)[(x-1)+2(x-2)] \\ &= (x-1)(3x-5) \end{aligned}$$

Intervals	$x<1$	$1<x<\frac{5}{3}$	$\frac{5}{3}<x$
Sign of y''	+	−	+
Behavior of y	Concave up	Concave down	Concave up

(a) There are no local maxima.

(b) There is a local (and absolute) minimum at $x=2$.

(c) There are points of inflection at $x=1$ and at $x=\frac{5}{3}$.

40. $y'=(x-1)\cdot(x-2)(x-4)$

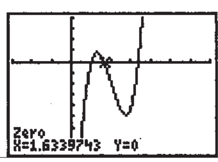
Intervals	$x<1$	$1<x<2$	$2<x<4$	$4<x$
Sign of y'	+	+	−	+
Behavior of y	Increasing	Increasing	Decreasing	Increasing

$$\begin{aligned} y'' &= \frac{d}{dx}[(x-1)^2(x^2-6x+8)] \\ &= (x-1)^2(2x-6)+(x^2-6x+8)(2)(x-1) \\ &= (x-1)[(x-1)(2x-6)+2(x^2-6x+8)] \\ &= (x-1)(4x^2-20x+22) \\ &= 2(x-1)(2x^2-10x+11) \end{aligned}$$

Note that the zeros of y'' are $x=1$ and

$$\begin{aligned} x &= \frac{10\pm\sqrt{10^2-4(2)(11)}}{4}=\frac{10\pm\sqrt{12}}{4} \\ &= \frac{5\pm\sqrt{3}}{2}\approx 1.63 \text{ or } 3.37. \end{aligned}$$

The zeros of y'' can also be found graphically, as shown.



[−3, 7] by [−8, 4]

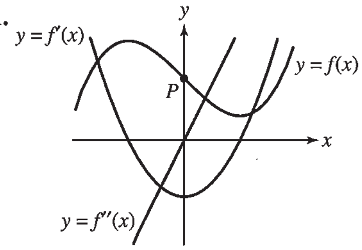
Intervals	$x<1$	$1<x<1.63$	$1.63<x<3.37$	$3.37<x$
Sign of y''	−	+	−	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

(a) Local maximum at $x=2$

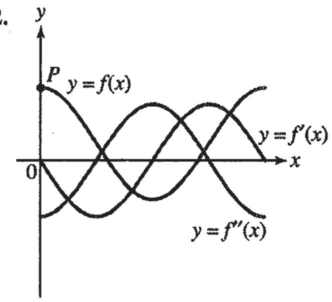
(b) Local minimum at $x=4$

(c) Points of inflection at $x=1$, at $x\approx 1.63$, and at $x\approx 3.37$.

41.



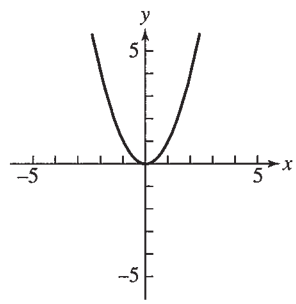
42.



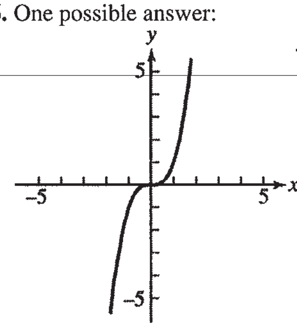
43. No. f must have a horizontal tangent at that point, but f could be increasing (or decreasing), and there would be no local extremum. For example, if $f(x)=x^3$, $f'(0)=0$ but there is no local extremum at $x=0$.

44. No. $f''(x)$ could still be positive (or negative) on both sides of $x=c$, in which case the concavity of the function would not change at $x=c$. For example, if $f(x)=x^4$, then $f''(0)=0$, but f has no inflection point at $x=0$.

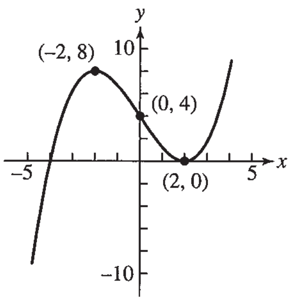
45. One possible answer:



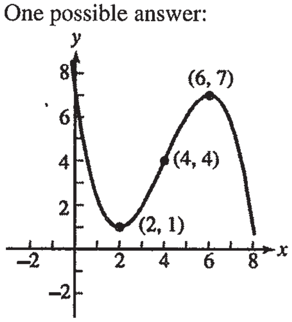
46. One possible answer:



47. One possible answer:



48. One possible answer:



49. **(a)** [0, 1], [3, 4], and [5.5, 6]

(b) [1, 3] and [4, 5.5]

(c) Local maxima: $x = 1, x = 4$ (if f is continuous at $x = 4$), and $x = 6$; local minima: $x = 0$, $x = 3$, and $x = 5.5$

50. If f is continuous on the interval [0, 3]:

(a) [0, 3]

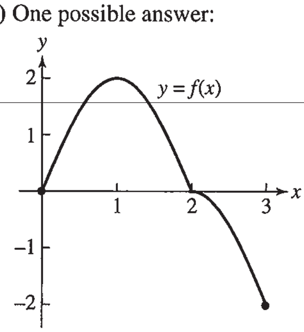
(b) Nowhere

(c) Local maximum: $x = 3$; local minimum: $x = 0$

51. **(a)** Absolute maximum at (1, 2); absolute minimum at (3, –2)

(b) None

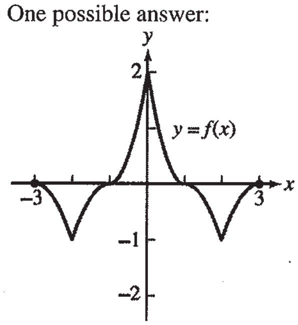
(c) One possible answer:



52. **(a)** Absolute maximum at (0, 2); absolute minimum at (2, –1) and (–2, –1)

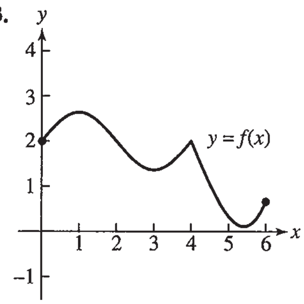
(b) At (1, 0) and (–1, 0)

(c) One possible answer:

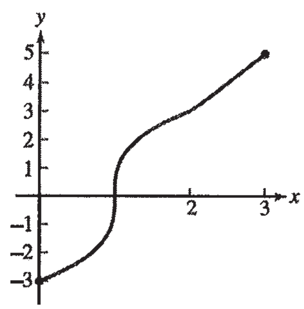


(d) Since f is even, we know $f(3) = f(-3)$. By the continuity of f, since $f(x) < 0$ when $2 < x < 3$, we know that $f(3) \le 0$, and since $f(2) = -1$ and $f'(x) > 0$ when $2 < x < 3$, we know that $f(3) > -1$. In summary, we know that $f(3) = f(-3), -1 < f(3) \le 0$, and $-1 < f(-3) \le 0$.

53.



54.



55. False. For example, consider $f(x) = x^4$ at $c = 0$.

56. True. This is the Second Derivative Test for a local maximum.

57. A. $y = ax^3 + 3x^2 + 4x + 5$ say $a = -2$

$$y' = -6x^2 + 6x + 4$$
$$y'' = -12x + 6$$
$$y'' = 0 \text{ at } \frac{1}{2}$$

Intervals	$x < 1/2$	$x > 1/2$
Sign of y''	+	–
Behavior of y	Concave up	Concave down

58. E.

59. C. $y = x^5 - 5x^4 + 3x + 7$

$$y' = 5x^4 - 20x^3 + 3$$
$$y'' = 20x^3 - 60x^2 = 20x^2(x-3)$$

Note that $y'' = 0$ at $x = 0$ and $x = 3$, but y'' only changes sign at $x = 3$.

Intervals	$x < 3$	$x > 3$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

$(3, -146)$ is an inflection point.

60. A. The slope of f' changes sign at $x = c$.

61. **(a)** In exercise 7, $a = 4$ and $b = 21$, so $-\dfrac{b}{3a} = -\dfrac{7}{4}$, which is the x-value where the point of inflection occurs. The local extrema are at $x = -2$ and $x = -\dfrac{3}{2}$, which are symmetric about $x = -\dfrac{7}{4}$.

(b) In exercise 2, $a = -2$ and $b = 6$, so $-\dfrac{b}{3a} = 1$, which is the x-value where the point of inflection occurs. The local extrema are at $x = 0$ and $x = 2$, which are symmetric about $x = 1$.

(c) $f'(x) = 3ax^2 + 2bx + c$ and
$f''(x) = 6ax + 2b$.
The point of inflection will occur where
$f''(x) = 0$, which is at $x = -\dfrac{b}{3a}$.
If there are local extrema, they will occur at the zeros of $f'(x)$. Since $f'(x)$ is quadratic, its graph is a parabola and any zeros will be symmetric about the vertex which will also be where $f''(x) = 0$.

62. **(a)** $$f'(x) = \frac{(1+ae^{-bx})(0) - (c)(-abe^{-bx})}{(1+ae^{-bx})^2}$$
$$= \frac{abce^{-bx}}{(1+ae^{-bx})^2} = \frac{abce^{+bx}}{(e^{bx}+a)^2},$$
so the sign of $f'(x)$ is the same as the sign of abc.

(b) $$f''(x) = \frac{(e^{bx}+a)^2(ab^2ce^{bx}) - (abce^{bx})2(e^{bx}+a)(be^{bx})}{(e^{bx}+a)^4}$$
$$= \frac{(e^{bx}+a)(ab^2ce^{bx}) - (abce^{bx})(2be^{bx})}{(e^{bx}+a)^3}$$
$$= -\frac{ab^2ce^{bx}(e^{bx}-a)}{(e^{bx}+a)^3}$$

Since $a > 0$, this changes sign when $x = \dfrac{\ln a}{b}$ due to the $e^{bx} - a$ factor in the numerator, and $f(x)$ has a point of inflection at the location.

63. **(a)** $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$
$f''(x) = 12ax^2 + 6bx + 2c$
Since $f''(x)$ is quadratic, it must have 0, 1, or 2 zeros. A quadratic with 0 or 1 zeros never changes sign, so f has no points of infection if $f''(x)$ has 0 or 1 zeros. If $f''(x)$ has 2 zeros, it will change sign twice, and $f(x)$ will have 2 points of inflection.

(b) Write a condition that must be satisfied by the coefficients if the graph of f has two points of inflection.

Quick Quiz Sections 4.1–4.3

1. (C) $f'(x) = 5(x-2)^4(x+3)^4 + 4(x-2)^5(x+3)^3 = 0$

$$x = -3, -\frac{7}{9}, 2$$

2. (D) $f'(x) = (x-3)^2 + 2(x-2)(x-3) = (x-3)(3x-7)$
$f'(x) = 0$ when $x = 3$ or $x = 7/3$
$f''(x) = 6x - 16$
$f''(3) = 2 > 0$
$f''(7/3) = -2 < 0$
Relative maximum is at $x = 7/3$ only.

3. (B) $x^2 - 9 = 0$
$x = \pm 3$
$f''(x) = 2x$
$f''(3) = 6 > 0$
$f''(-3) = -6 < 0$
$f'(x) = (x^2 - 9)g(x)$; where $g(x) < 0$ for all x. Thus the sign graph for $f'(x)$ looks like this:

$-$ $+$ $-$

-3 3

By the First Derivative Test, f has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.

4. **(a)** $$\frac{d}{dx}\left(3\ln(x^2+2) - 2x\right) = 3\frac{2x}{x^2+2} - 2 = 0$$
$$x = 1, 2$$

Intervals	$-2 < x < 1$	$1 < x < 2$	$2 < x < 4$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

f has relative minima at $x = 1$ and $x = 4$, f has relative maxima at $x = \pm 2$

(b) $$f''(x) = \frac{d}{dx}\left(\frac{6x}{x^2+2} - 2\right)$$
$$f''(x) = \frac{6}{x^2+2} - \frac{12x^2}{(x^2+2)^2} = 0$$
$$x = \pm\sqrt{2}$$

f has points of inflection at $x = \pm\sqrt{2}$

(c) The absolute maximum is at $x = -2$ and $f(x) = 3\ln 6 + 4$.