4. $h'(x) = e^{2x} \cdot \frac{d}{dx} 2x = 2e^{2x}$

5. Graph (c), since this is the only graph that has positive slope at c.

6. Graph (b), since this is the only graph that represents a differentiable function at a and b and has negative slope at c.

7. Graph (d), since this is the only graph representing a function that is differentiable at b but not at a.

8. Graph (a), since this is the only graph that represents a function that is not differentiable at a or b.

9. As $x \to 3^-$, $\sqrt{9-x^2} \to 0^+$. Therefore, $\lim_{x\to 3^-} f(x) = \infty$.

10. As $x \to 3^+$, $\sqrt{9-x^2} \to 0^+$. Therefore, $\lim_{x\to -3^+} f(x) = \infty$.

11. (a) $\frac{d}{dx}(x^3 - 2x) = 3x^2 - 2$

$f'(1) = 3(1)^2 - 2 = 1$

(b) $\frac{d}{dx}(x+2) = 1$

$f'(3) = 1$

(c) Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{[(2+h)^3 - 2(2+h)] - 4}{h}$$
$$= \lim_{h\to 0^-} \frac{h^3+6h^2+10h}{h} = \lim_{h\to 0^-} (h^2+6h+10)$$
$$= 10$$

Right-hand derivative:

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{[(2+h)+2]-4}{h}$$
$$= \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1$$
$$= 1$$

Since the left-and right-hand derivatives are not equal, $f'(2)$ is undefined.

12. (a) The domain is $x \neq 2$. (See the solution for 11.(c)).

(b) $f'(x) = \begin{cases} 3x^2 - 2, & x < 2 \\ 1, & x > 2 \end{cases}$

Section 4.1 Exercises

1. Minima at (–2, 0) and (2, 0), maximum at (0, 2)
2. Local minimum at (–1, 0), local maximum at (1, 0)
3. Maximum at (0, 5) Note that there is no minimum since the endpoint (2, 0) is excluded from the graph.
4. Local maximum at (–3, 0), local minimum at (2, 0), maximum at (1, 2), minimum at (0, –1)
5. Maximum at $x = b$, minimum at $x = c_2$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.
6. Maximum at $x = c$, minimum at $x = b$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.
7. Maximum at $x = c$, no minimum;
The Extreme Value Theorem does not apply, because the function is not defined on a closed interval.
8. No maximum, no minimum;
The Extreme Value Theorem does not apply, because the function is not continuous or defined on a closed interval.
9. Maximum at $x = c$, minimum at $x = a$;
The Extreme Value Theorem does not apply, because the function is not continuous.
10. Maximum at $x = a$, minimum at $x = c$;
The Extreme Value Theorem does not apply since the function is not continuous.
11. The first derivative $f'(x) = -\frac{1}{x^2} + \frac{1}{x}$ has a zero at $x = 1$.

Critical point value: $f(1) = 1 + \ln 1 = 1$

Endpoint values: $f(0.5) = 2 + \ln 0.5 = 1.307$

$$f(4) = \frac{1}{4} + \ln 4 = 1.636$$

Maximum value is $\frac{1}{4} + \ln 4$ at $x = 4$;

minimum value is 1 at $x = 1$;

local maximum at $\left(\frac{1}{2}, 2 - \ln 2\right)$

12. The first derivative $g'(x) = -e^{-x}$ has no zeros, so we need only consider the endpoints.

$g(-1) = e^{-(-1)} = e \quad g(1) = e^{-1} = \frac{1}{e}$

Maximum value is e at $x = -1$;

minimum value is $\frac{1}{e}$ at $x = 1$.

13. The first derivative $h'(x) = \frac{1}{x+1}$ has no zeros, so we need only consider the endpoints.

$h(0) = \ln 1 = 0 \qquad h(3) = \ln 4$

Maximum value is $\ln 4$ at $x = 3$;

minimum value is 0 at $x = 0$.

14. The first derivative $k'(x) = -2xe^{-x^2}$ has a zero at $x = 0$. Since the domain has no endpoints, any extreme value must occur at $x = 0$. Since $k(0) = e^{-0^2} = 1$ and $\lim_{x\to\pm\infty} k(x) = 0$, the maximum value is 1 at $x = 0$.

15. The first derivative $f'(x) = \cos\left(x + \frac{\pi}{4}\right)$, has zeros at $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$.

15. Continued

Critical point values: $x = \frac{\pi}{4}$ $\quad f(x) = 1$

$x = \frac{5\pi}{4}$ $\quad f(x) = -1$

Endpoint values: $x = 0$ $\quad f(x) = \frac{1}{\sqrt{2}}$

$x = \frac{7\pi}{4}$ $\quad f(x) = 0$

Maximum value is 1 at $x = \frac{\pi}{4}$;

minimum value is -1 at $x = \frac{5\pi}{4}$;

local minimum at $\left(0, \frac{1}{\sqrt{2}}\right)$;

local maximum at $\left(\frac{7\pi}{4}, 0\right)$

16. The first derivative $g'(x) = \sec x \tan x$ has zeros at $x = 0$ and $x = \pi$ and is undefined at $x = \frac{\pi}{2}$.

Since $g(x) = \sec x$ is also undefined at $x = \frac{\pi}{2}$, the critical points occur only at $x = 0$ and $x = \pi$.

Critical point values: $x = 0$ $\quad g(x) = 1$

$x = \pi$ $\quad g(x) = -1$

Since the range of $g(x)$ is $(-\infty, -1] \cup [1, \infty)$, these values must be a local minimum and local maximum, respectively.

Local minimum at $(0, 1)$; local maximum at $(\pi, -1)$

17. The first derivative $f'(x) = \frac{2}{5}x^{-3/5}$ is never zero but is undefined at $x = 0$.

Critical point value: $x = 0$ $\quad f(x) = 0$

Endpoint value: $x = -3$ $\quad f(x) = (-3)^{2/5}$

$= 3^{2/5} = 1.552$

Since $f(x) > 0$ for $x \neq 0$, the critical point at $x = 0$ is a local minimum, and since $f(x) \leq (-3)^{2/5}$ for $-3 \leq x < 1$, the endpoint value at $x = -3$ is a global maximum.

Maximum value is $3^{2/5}$ at $x = -3$;

minimum value is 0 at $x = 0$.

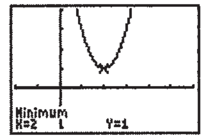
18. The first derivative $f'(x) = \frac{3}{5}x^{-2/5}$ is never zero but is undefined at $x = 0$.

Critical point value: $x = 0$ $\quad f(x) = 0$

Endpoint value: $x = 3$ $\quad f(x) = 3^{3/5} = 1.933$

Since $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$, the critical point is not a local minimum or maximum. The maximum value is $3^{3/5}$ at $x = 3$.

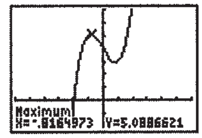
19.



$[-2, 6]$ by $[-2, 4]$

Minimum value is 1 at $x = 2$.

20.



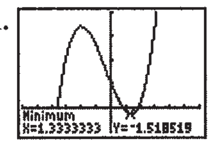
$[-6, 6]$ by $[-2, 7]$

To find the exact values, note that $y' = 3x^2 - 2$, which is zero when $x = \pm\sqrt{\frac{2}{3}}$. Local maximum at

$\left(-\sqrt{\frac{2}{3}}, 4 + \frac{4\sqrt{6}}{9}\right) \approx (-0.816, 5.089)$; local minimum at

$\left(\frac{\sqrt{2}}{3}, 4 - \frac{4\sqrt{6}}{9}\right) \approx (0.816, 2.911)$

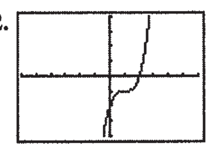
21.



$[-6, 6]$ by $[-5, 20]$

To find the exact values, note that $y' = 3x^2 + 2x - 8 = (3x - 4)(x + 2)$, which is zero when $x = -2$ or $x = \frac{4}{3}$. Local maximum at $(-2, 17)$; local minimum at $\left(\frac{4}{3}, -\frac{41}{27}\right)$

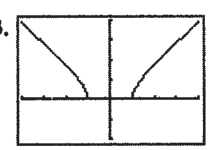
22.



$[-6, 6]$ by $[-4, 4]$

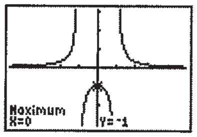
Note that $y' = 3x^2 - 6x + 3 = 3(x - 1)^2$, which is zero at $x = 1$. The graph shows that the function assumes lower values to the left and higher values to the right of this point, so the function has no local or global extreme values.

23.



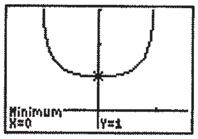
$[-4, 4]$ by $[-2, 4]$

Minimum value is 0 at $x = -1$ and at $x = 1$.

24. 

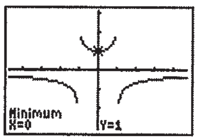
[−4.7, 4.7] by [−3.1, 3.1]

To confirm that there are no "hidden" extrema, note that $y' = -(x^2-1)^{-2}(2x) = \dfrac{-2x}{(x^2-1)^2}$ which is zero only at $x = 0$ and is undefined only where y is undefined. There is a local maximum at $(0, -1)$.

25. 

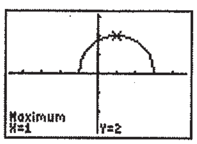
[−1.5, 1.5] by [−0.5, 3]

The minimum value is 1 at $x = 0$.

26. 

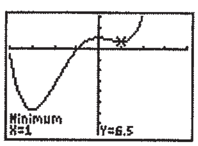
[−4.7, 4.7] by [−3.1, 3.1]

The actual graph of the function has asymptotes at $x = \pm 1$, so there are no extrema near these values. (This is an example of *grapher failure.*) There is a local minimum at $(0, 1)$.

27. 

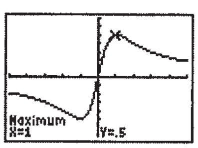
[−4.7, 4.7] by [−3.1, 3.1]

Maximum value is 2 at $x = 1$;
minimum value is 0 at $x = -1$ and at $x = 3$.

28. 

[−4, 4] by [−80, 30]

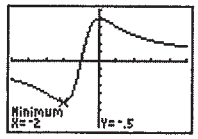
Minimum value is $-\dfrac{115}{2}$ at $x = -3$;
local maximum at $(0, 10)$;
local minimum at $\left(1, \dfrac{13}{2}\right)$

29. 

[−5, 5] by [−0.7, 0.7]

Maximum value is $\dfrac{1}{2}$ at $x = 1$;

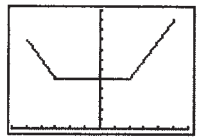
minimum value is $-\dfrac{1}{2}$ at $x = -1$.

30. 

[−5, 5] by [−0.8, 0.6]

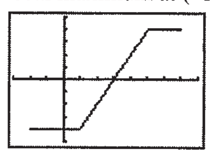
Maximum value is $\dfrac{1}{2}$ at $x = 0$;

minimum value is $-\dfrac{1}{2}$ at $x = -2$.

31. 

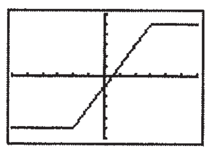
[−6, 6] by [0, 12]

Maximum value is 11 at $x = 5$;
minimum value is 5 on the interval $[-3, 2]$;
local maximum at $(-5, 9)$

32. 

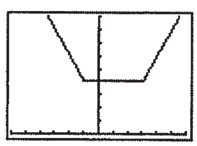
[−3, 8] by [−5, 5]

Maximum value is 4 on the interval $[5, 7]$;
minimum value is −4 on the interval $[-2, 1]$.

33. 

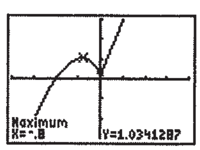
[−6, 6] by [−6, 6]

Maximum value is 5 on the interval $[3, \infty)$;
minimum value is −5 on the interval $(-\infty, -2]$.

34. 

[−6, 6] by [0, 9]

Minimum value is 4 on the interval $[-1, 3]$

35. 

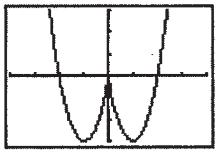
[−4, 4] by [−3, 3]

$$y' = x^{2/3}(1) + \frac{2}{3}x^{-1/3}(x+2) = \frac{5x+4}{3\sqrt[3]{x}}$$

35. Continued

crit. pt.	Derivative	extremum	value
$x=-\frac{4}{5}$	0	local max	$\frac{12}{25}10^{1/3}\approx 1.034$
$x=0$	undefined	local min	0

36.

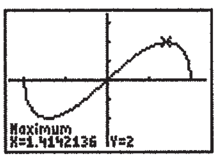


[−4, 4] by [−3, 3]

$$y'=x^{2/3}(2x)+\frac{2}{3}x^{-1/3}(x^2-4)=\frac{8x^2-8}{3\sqrt[3]{x}}$$

crit. pt.	derivative	extremum	value
$x=-1$	0	minimum	−3
$x=0$	undefined	local max	0
$x=1$	0	minumum	−3

37.

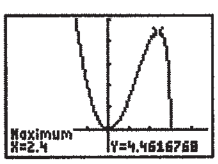


[−2.35, 2.35] by [−3.5, 3.5]

$$y'=x\cdot\frac{1}{2\sqrt{4-x^2}}(-2x)+(1)\sqrt{4-x^2}$$
$$=\frac{-x^2+(4-x^2)}{\sqrt{4-x^2}}=\frac{4-2x^2}{\sqrt{4-x^2}}$$

crit. pt.	derivative	extremum	value
$x=-2$	undefined	local max	0
$x=-\sqrt{2}$	0	minimum	−2
$x=\sqrt{2}$	0	maximum	2
$x=2$	undefined	local min	0

38.

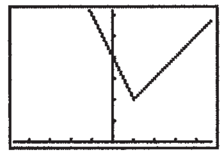


[−4.7, 4.7] by [−1, 5]

$$y=x^2\cdot\frac{1}{2\sqrt{3-x}}(-1)+2x\sqrt{3-x}$$
$$=\frac{-x^2+4x(3-x)}{2\sqrt{3-x}}=\frac{-5x^2+12x}{2\sqrt{3-x}}$$

crit. pt.	derivative	extremum	value
$x=0$	0	minimum	0
$x=\frac{12}{5}$	0	local max	$\frac{144}{125}15^{1/2}\approx 4.462$
$x=3$	undefined	minimum	0

39.

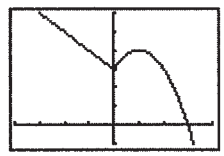


[−4.7, 4.7] by [0, 6.2]

$$y'=\begin{cases}-2, & x<1\\ 1, & x>1\end{cases}$$

crit. pt.	derivative	extremum	value
$x=1$	undefined	minimum	2

40.

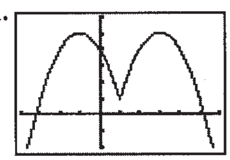


[−4, 4] by [−1, 6]

$$y'=\begin{cases}-1, & x<0\\ 2-2x, & x>0\end{cases}$$

crit. pt.	derivative	extremum	value
$x=0$	undefined	local min	3
$x=1$	0	local max	4

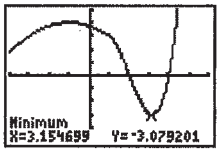
41.



[−4, 6] by [−2, 6]

$$y'=\begin{cases}-2x-2, & x<1\\ -2x+6, & x>1\end{cases}$$

crit. pt.	derivative	extremum	value
$x=-1$	0	maximum	5
$x=1$	undefined	local min	1
$x=3$	0	maximum	5

42. 

$[-4, 6]$ by $[-5, 5]$

We begin by determining whether $f'(x)$ is defined at $x = 1$, where

$$f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$$

Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{-\frac{1}{4}(1+h)^2 - \frac{1}{2}(1+h) + \frac{15}{4} - 3}{h}$$
$$= \lim_{h \to 0^-} \frac{-h^2 - 4h}{4h}$$
$$= \lim_{h \to 0^-} \frac{1}{4}(-h-4)$$
$$= -1$$

Right-hand derivative:

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0^+} \frac{(1+h)^3 - 6(1+h)^2 + 8(1+h) - 3}{h}$$
$$= \lim_{h \to 0^+} \frac{h^3 - 3h^2 - h}{h}$$
$$= \lim_{h \to 0^+} (h^2 - 3h - 1)$$
$$= -1$$

Thus $f'(x) = \begin{cases} -\frac{1}{2}x - \frac{1}{2}, & x \le 1 \\ 3x^2 - 12x + 8, & x > 1 \end{cases}$

Note that $-\frac{1}{2}x - \frac{1}{2} = 0$ when $x = -1$, and

$3x^2 - 12x + 8 = 0$ when $x = \dfrac{12 \pm \sqrt{12^2 - 4(3)(8)}}{2(3)}$

$= \dfrac{12 \pm \sqrt{48}}{6} = 2 \pm \dfrac{2\sqrt{3}}{3}.$

But $2 - \dfrac{2\sqrt{3}}{3} \approx 0.845 < 1$, so the only critical points occur at $x = -1$ and $x = 2 + \dfrac{2\sqrt{3}}{3} \approx 3.155$.

crit. pt.	derivative	extremum	value
$x = -1$	0	local max	4
$x \approx 3.155$	0	local min	≈ -3.079

43. (a) $V(x) = 160x - 52x^2 + 4x^3$

$V'(x) = 160 - 104x + 12x^2 = 4(x-2)(3x-20)$

The only critical point in the interval $(0, 5)$ is at $x = 2$. The maximum value of $V(x)$ is 144 at $x = 2$.

(b) The largest possible volume of the box is 144 cubic units, and it occurs when $x = 2$.

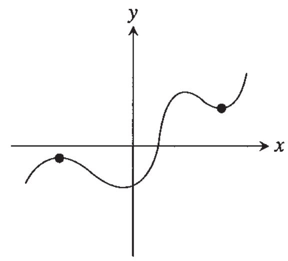
44. (a) $P'(x) = 2 - 200x^{-2}$

The only critical point in the interval $(0, \infty)$ is at $x = 10$. The minimum value of $P(x)$ is 40 at $x = 10$.

(b) The smallest possible perimeter of the rectangle is 40 units and it occurs at $x = 10$, which makes the rectangle a 10 by 10 square.

45. False. For example, the maximum could occur at a corner, where $f'(c)$ would not exist.

46. False. Consider the graph below.



47. E. $\dfrac{d}{dx}(4x - x^2 + 6) = 4 - 2x$

$4 - 2x = 0$

$x = 2$

$f(2) = 4(2) - (2)^2 + 6 = 10$

48. E. See Theorem 2.

49. B. $\dfrac{d}{dx}(x^3 - 6x + 5) = 3x^2 - 6$

$3x^2 - 6 = 0$

$x = \pm\sqrt{2}$

50. B.

51. (a) No, since $f'(x) = \frac{2}{3}(x-2)^{-1/3}$, which is undefined at $x = 2$.

(b) The derivative is defined and nonzero for all $x \ne 2$. Also, $f(2) = 0$ and $f(x) > 0$ for all $x \ne 2$.

(c) No, $f(x)$ need not have a global maximum because its domain is all real numbers. Any restriction of f to a closed interval of the form $[a, b]$ would have both a maximum value and a minimum value on the interval.

(d) The answers are the same as (a) and (b) with 2 replaced by a.

52. Note that $f(x) = \begin{cases} -x^3 + 9x, & x \le -3 \text{ or } 0 \le x < 3 \\ x^3 - 9x, & -3 < x < 0 \text{ or } x \ge 3. \end{cases}$

Therefore, $f'(x) = \begin{cases} -3x^2 + 9, & x < -3 \text{ or } 0 < x < 3 \\ 3x^2 - 9, & -3 < x < 0 \text{ or } x > 3. \end{cases}$

(a) No, since the left- and right-hand derivatives at $x = 0$ are -9 and 9, respectively.

52. Continued

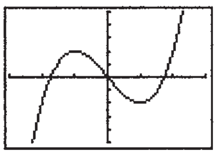
(b) No, since the left- and right-hand derivatives at $x = 3$ are -18 and 18, respectively.

(c) No, since the left- and right-hand derivatives at $x = -3$ are -18 and 18, respectively.

(d) The critical points occur when $f'(x) = 0$ (at $x = \pm\sqrt{3}$) and when $f'(x)$ is undefined (at $x = 0$ or $x = \pm 3$). The minimum value is 0 at $x = -3$, at $x = 0$, and at $x = 3$; local maxima occur at $(-\sqrt{3}, 6\sqrt{3})$ and $(\sqrt{3}, 6\sqrt{3})$.

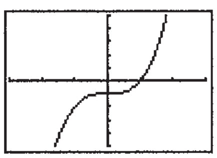
53. (a) $f'(x) = 3ax^2 + 2bx + c$ is a quadratic, so it can have 0, 1, or 2 zeros, which would be the critical points of f.

Examples:



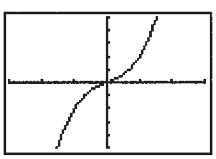
$[-3, 3]$ by $[-5, 5]$

The function $f(x) = x^3 - 3x$ has two critical points at $x = -1$ and $x = 1$.



$[-3, 3]$ by $[-5, 5]$

The function $f(x) = x^3 - 1$ has one critical point at $x = 0$.



$[-3, 3]$ by $[-5, 5]$

The function $f(x) = x^3 + x$ has no critical points.

(b) The function can have either two local extreme values or no extreme values. (If there is only one critical point, the cubic function has no extreme values.)

54. (a) By the definition of local maximum value, there is an open interval containing c where $f(x) \le f(c)$, so $f(x) - f(c) \le 0$.

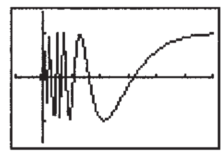
(b) Because $x \to c^+$, we have $(x - c) > 0$, and the sign of the quotient must be negative (or zero). This means the limit is nonpositive.

(c) Because $x \to c^-$, we have $(x - c) < 0$, and the sign of the quotient must be positive (or zero). This means the limit is nonnegative.

(d) Assuming that $f'(c)$ exists, the one-sided limits in (b) and (c) above must exist and be equal. Since one is nonpositive and one is nonnegative, the only possible common value is 0.

(e) There will be an open interval containing c where $f(x) - f(c) \ge 0$. The difference quotient for the left-hand derivative will have to be negative (or zero), and the difference quotient for the right-hand derivative will have to be positive (or zero). Taking the limit, the left-hand derivative will be nonpositive, and the right-hand derivative will be nonnegative. Therefore, the only possible value for $f'(c)$ is 0.

55. (a)



$[-0.1, 0.6]$ by $[-1.5, 1.5]$

$f(0) = 0$ is not a local extreme value because in any open interval containing $x = 0$, there are infinitely many points where $f(x) = 1$ and where $f(x) = -1$.

(b) One possible answer, on the interval $[0, 1]$:

$$f(x) = \begin{cases} (1-x)\cos\dfrac{1}{1-x}, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

This function has no local extreme value at $x = 1$. Note that it is continuous on $[0, 1]$.

Section 4.2 Mean Value Theorem (pp. 196–204)

Quick Review 4.2

1. $2x^2 - 6 < 0$

$$2x^2 < 6$$
$$x^2 < 3$$
$$-\sqrt{3} < x < \sqrt{3}$$

Interval: $(-\sqrt{3}, \sqrt{3})$

2. $3x^2 - 6 > 0$

$$3x^2 > 6$$
$$x^2 > 2$$
$$x < -\sqrt{2} \text{ or } x > \sqrt{2}$$

Intervals: $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

3. Domain: $8 - 2x^2 \ge 0$

$$8 \ge 2x^2$$
$$4 \ge x^2$$
$$-2 \le x \le 2$$

The domain is $[-2, 2]$.

4. f is continuous for all x in the domain, or, in the interval $[-2, 2]$.